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HIGHER ORDER STATISTICS AND MULTIVARIATE VECTOR HERMITE POLYNOMIALS FOR NONLINEAR ANALYSIS OF MULTIDIMENSIONAL TIME SERIES

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Abstract. In this work we show that it is possible to generalize the definitions for both the cumulants and Hermite polynomials for vector valued variables such that the expressions remain quite elementary and transparent. We apply a particular differential operator recursively to get the appropriate definitions. The basic properties of the Kronecker cumulants are listed using Kronecker products and commutation matrices. The most important formulae for multivariate Hermite polynomials with vector values are proved. The Kronecker cumulants for Gaussian vector system and Hermite polynomials are given. The definition of multiple Wiener–Itô integral for vector valued stationary flows and the chaotic representation of stationary subordinated vector processes are also considered.

1. Introduction

Cumulants, spectra and Hermite polynomials are basic tools for the statistical analysis of nonlinear time series. The applications of these methods include testing for Gaussianity and linearity, estimating parameters for non-Gaussian models, [2], [21], [19], [25], [24], [4], modelling non-Gaussian random fields [16] (for a general treatment of homogeneous and isotropic random fields see [28]) and so on. Although Brillinger gives the general definition of cumulants for multiple time series the expressions are usually given in coordinatewise manner. Recently in applications, [12], [22], there has been some effort done for using vector notation. The theory of Hermite polynomials has some long history starting from the beginning of the last century, see [23] and references therein, up to nowadays, [27], [11], [26], [3] for instance. The generalizations of Hermite polynomials have reached the cases of multiple scalar valued and of single d-variate vector valued case, [8]. There are further results of this last one for abstract spaces [10], [7] as well.

In this work we show that it is possible to generalize the definitions of both the cumulants and Hermite polynomials for vector valued variables (with not necessarily same dimensions) such that the expressions remain quite elementary and transparent. The idea is that keeping the methods given in the book [15] in mind we apply a particular differential operator recursively to get the appropriate definitions. We list the basic properties of the Kronecker cumulants in terms of matrix notations using extensively the well known results concerning to the Kronecker products and commutation matrices, see [13] for details. The most important formulae for multivariate Hermite polynomials with vector values are proved. We spare some space listing the Kronecker cumulants for Gaussian vector system and Hermite polynomials without proof. Finally we give the definition of multiple Wiener–Itô integral for vector valued stationary flows in the

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space of $\mathfrak{L}^2$ functionals of stationary Gaussian processes. The chaotic representation of stationary subordinated vector processes are also considered.

2. Kronecker cumulants

2.1. Kronecker moments. Suppose that the vector $\underline{X}$ is partitioned into two parts $\underline{X} = [\underline{X}_1', \underline{X}_2']'$ with dimensions $[d_1, d_2]$ which are not necessary same. The characteristic function of $\underline{X}$ is written

$$\begin{aligned}\varphi_{\underline{X}}(\underline{\lambda}) &= \varphi_{\underline{X}_1,\underline{X}_2}(\underline{\lambda}_1,\underline{\lambda}_2) = \mathsf{E}\exp(i[(\underline{\lambda}_1,\underline{X}_1)+(\underline{\lambda}_2,\underline{X}_2)]) \\ &= \sum_{k,l=0}^{\infty}\frac{i^{k+l}}{k!\,l!}\,\mathsf{E}(\underline{\lambda}_1,\underline{X}_1)^k(\underline{\lambda}_2,\underline{X}_{21})^l \\ &= \sum_{k,l=0}^{\infty}\frac{i^{k+l}}{k!\,l!}\,\mathsf{E}\big(\underline{X}_1'^{\otimes k}\otimes\underline{X}_2'^{\otimes l}\big)\big(\underline{\lambda}_1^{\otimes k}\otimes\underline{\lambda}_2^{\otimes l}\big),\end{aligned}$$

where $\underline{\lambda}$ is partitioned according to $\underline{X}$ into two parts $\underline{\lambda}' = [\underline{\lambda}_1', \underline{\lambda}_2']$ and

$$(\underline{\lambda},\underline{X}) = \sum_{j=1}^{n}\lambda_j X_j.$$

Now the operator $D^{\otimes 2}_{\underline{\lambda}_1,\underline{\lambda}_2} = D^{\otimes}_{\underline{\lambda}_2}D^{\otimes}_{\underline{\lambda}_1}$, see Appendix B, results on φ

$$(-i)^2 D^{\otimes 2}_{\underline{\lambda}_1,\underline{\lambda}_2}\varphi_{\underline{X}_1,\underline{X}_2}(\underline{\lambda}_1,\underline{\lambda}_2)\Big|_{\boldsymbol{\lambda}_{(1:2)}=0} = \mathsf{E}(\underline{X}_1\otimes\underline{X}_2).$$

Therefore the following definition is a generalization of the scalar valued case, assuming the appropriate differentiability of φ. Suppose $\underline{X}_{(1:n)} = [\underline{X}_1', \underline{X}_2', \dots, \underline{X}_n']'$, i.e., the vector $\underline{X}$ is partitioned with dimensions $[d_1, d_2, \dots, d_n]$. The definition of the Kronecker moment is

$$\begin{aligned}\mathsf{E}(\underline{X}_1\otimes\underline{X}_2\cdots\otimes\underline{X}_n) &= \mathsf{E}\prod\nolimits_{j=1}^{\otimes n}\underline{X}_j \\ &= (-i)^n D^{\otimes n}_{\underline{\lambda}_1,\underline{\lambda}_2,\dots,\underline{\lambda}_n}\varphi_{\underline{X}_1,\underline{X}_2,\dots,\underline{X}_n}(\underline{\lambda}_1,\underline{\lambda}_2,\dots,\underline{\lambda}_n)\Big|_{\boldsymbol{\lambda}_{(1:n)}=0}.\end{aligned}$$

Note here that the order in the product and the derivative are important, the Kronecker moment is not symmetric if the variables $\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n$ are different.

2.2. Definition of Kronecker cumulants. Consider the first order derivatives at zero of the logarithm of the characteristic function of $\underline{X} = (X_1, X_2, \dots, X_n)'$, i.e.,

$$(-i)^n\frac{\partial^n \ln\varphi_{\underline{X}}(\underline{\lambda})}{\partial\underline{\lambda}^{\mathbf{1}_{[n]}}}\Bigg|_{\lambda_{(1:n)}=0} = \operatorname{Cum}_n(\underline{X}),$$

where $\mathbf{1}_{[n]}$ denotes a row vector having all ones in its coordinates, i.e., $\mathbf{1}_{[n]} = (1, 1, \dots, 1)$ with dimension n. The cumulant$\operatorname{Cum}_n(\underline{X})$ is defined as the the first order, by each of its variable, derivative at zero of $\ln\varphi_{\underline{X}}$ This definition is correct even some components of $\underline{X}$ were the same. That formula allows us to act as if all the components of the vector $\underline{X}$ were different so it is satisfactory to define the cumulant for different variables only. The entries of $\underline{X}$ are scalar variables.

Now consider

$$\boldsymbol{X}_{(1:n)} = \big(\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n\big),$$

as a series of vectors with dimensions $d_{(1:n)} = (d_1, d_2, \dots, d_n)$. The corresponding characteristic function is

$$\varphi_{\boldsymbol{X}_{(1:n)}}\big(\boldsymbol{\lambda}_{(1:n)}\big) = \varphi_{\operatorname{Vec}\boldsymbol{X}_{(1:n)}}\big(\operatorname{Vec}\boldsymbol{\lambda}_{(1:n)}\big) = \mathsf{E}\exp\big(i(\operatorname{Vec}\boldsymbol{\lambda}_{(1:n)}, \operatorname{Vec}\boldsymbol{X}_{(1:n)})\big),$$

where $\boldsymbol{\lambda}_{(1:n)} = (\underline{\lambda}_1, \underline{\lambda}_2, \dots, \underline{\lambda}_n)$ is a corresponding series of vectors with dimensions $d_{(1:n)} = (d_1, d_2, \dots, d_n)$, see Appendix A for the definition of Vec operator. The the logarithm of the characteristic function $\varphi_{\operatorname{Vec}\boldsymbol{X}_{(1:n)}}(\operatorname{Vec}\boldsymbol{\lambda}_{(1:n)})$ will be called cumulant function, i.e.,

$$\phi_{\operatorname{Vec}\boldsymbol{X}_{(1:n)}}\big(\operatorname{Vec}\boldsymbol{\lambda}_{(1:n)}\big) = \ln \varphi_{\operatorname{Vec}\boldsymbol{X}_{(1:n)}}\big(\operatorname{Vec}\boldsymbol{\lambda}_{(1:n)}\big).$$

There will be no confusion if we omit the Vec operator and use $\phi_{\boldsymbol{X}_{(1:n)}}(\boldsymbol{\lambda}_{(1:n)})$ for $\phi_{\operatorname{Vec}\boldsymbol{X}_{(1:n)}}(\operatorname{Vec}\underline{\boldsymbol{\lambda}}_{(1:n)})$. The first order derivative by each vector variable at zero of the cumulant function $\phi_{\boldsymbol{X}_{(1:n)}}(\boldsymbol{\lambda}_{(1:n)})$ according to $\boldsymbol{\lambda}_{(1:n)}$ is defined as cumulant of $\boldsymbol{X}_{(1:n)}$. More precisely apply the operator $D^{\otimes n}_{\boldsymbol{\lambda}_{(1:n)}}\phi = D^{\otimes}_{\underline{\lambda}_n}(D^{\otimes n-1}_{\boldsymbol{\lambda}_{(1:n-1)}}\phi)$, recursively and the result is a column vector of the partial differentials of order n, which is first order by each variable $\underline{\lambda}_j$. The dimension of $D^{\otimes n}_{\boldsymbol{\lambda}_{(1:n)}}$ is $d^{\mathbf{1}[n]}_{1:n} = \prod_{j=1}^{n} d_j$. Now the definition of the nth order cumulant of vectors $\boldsymbol{X}_{(1:n)}$ is

$$\underline{\mathrm{Cum}}_n(\boldsymbol{X}_{(1:n)}) = (-i)^n D^{\otimes n}_{\boldsymbol{\lambda}_{(1:n)}} \phi_{\boldsymbol{X}_{(1:n)}}(\boldsymbol{\lambda}_{(1:n)})\Big|_{\boldsymbol{\lambda}_{(1:n)}=0}. \tag{1}$$

Therefore $\underline{\mathrm{Cum}}_n(\boldsymbol{X}_{(1:n)})$ is a vector of dimension $d^{\mathbf{1}[n]}_{1:n}$ having all possible cumulants of entries of vectors $\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n$ by the order defined by the Kronecker product.

Example 1. The cumulant function of $\boldsymbol{X}_{(1,2)}$ is

$$\psi_{\operatorname{Vec}\boldsymbol{X}_{(1,2)}}(\underline{u}_1, \underline{u}_2) = i\big(\underline{\mu}_1'\underline{u}_1 + \underline{\mu}_2'\underline{u}_2\big) - \frac{1}{2}\begin{bmatrix}\underline{u}_1\\ \underline{u}_2\end{bmatrix}' C_{(\underline{X}_1', \underline{X}_2')'}\begin{bmatrix}\underline{u}_1\\ \underline{u}_2\end{bmatrix},$$

where the covariance matrix $C_{(\underline{X}_1', \underline{X}_2')'}$ is partitioned as

$$C_{(\underline{X}_1', \underline{X}_2')'} = \begin{bmatrix} C_{1,1} & C_{1,2}\\ C_{2,1} & C_{2,2}\end{bmatrix},$$

and the qratic form is written by

$$\begin{bmatrix}\underline{u}_1\\ \underline{u}_2\end{bmatrix}' C_{(\underline{X}_1', \underline{X}_2')'}\begin{bmatrix}\underline{u}_1\\ \underline{u}_2\end{bmatrix} = \underline{u}_1' C_{1,1}\underline{u}_1 + \underline{u}_1' C_{1,2}\underline{u}_2 + \underline{u}_2' C_{2,1}\underline{u}_1 + \underline{u}_2' C_{2,2}\underline{u}_2.$$

Now the first order cumulant is

$$\underline{\mathrm{Cum}}_1(\underline{X}_j) = \underline{\mu}_j,$$

and it is clear that any higher order cumulant then 2 is zero. It is easy to see that the second order cumulants are the vectors of the covariance matrices, i.e.,

$$\underline{\mathrm{Cum}}_2(\underline{X}_j, \underline{X}_k) = \operatorname{Vec} C_{k,j}, \qquad j, k = 1, 2.$$

In general by formula (1) we obtain

$$\underline{\mathrm{Cum}}_1(\underline{X}_t) = \mathsf{E}\,\underline{X}_t, \qquad t = 1, 2;$$

and

$$\begin{aligned}\underline{\mathrm{Cum}}_2\big(\underline{X}_{t_1}, \underline{X}_{t_2}\big) &= \mathsf{E}\big[(\underline{X}_{t_1} - \mathsf{E}\,\underline{X}_{t_1}) \otimes (\underline{X}_{t_2} - \mathsf{E}\,\underline{X}_{t_2})\big]\\ &= \operatorname{Vec}\underline{\underline{\mathrm{Cov}}}\big(\underline{X}_{t_2}, \underline{X}_{t_1}\big), \qquad t_1, t_2 = 1, 2,\end{aligned} \tag{2}$$

where $\underline{\underline{\mathrm{Cov}}}(\underline{X}_{t_2}, \underline{X}_{t_1})$ denotes the covariance matrix of the vectors $\underline{X}_{t_2}$ and $\underline{X}_{t_1}$.

2.2.1. Basic properties. Suppose for simplicity that dimensions of $\underline{X}_1, \underline{X}_2, \dots, \underline{X}_n$ are same, say d.

1. Symmetry. If $d > 1$ then the cumulants are not symmetric but fulfils the equation
$$\underline{\operatorname{Cum}}_n\big(\boldsymbol{X}_{(1:n)}\big) = K^{-1}_{\mathfrak{p}(1:n)}\big(d_{[n]}\big)\underline{\operatorname{Cum}}_n\big(\boldsymbol{X}_{\mathfrak{p}(1:n)}\big),$$
where
$$\mathfrak{p}(1:n) = (\mathfrak{p}(1), \mathfrak{p}(2), \dots, \mathfrak{p}(n)), \qquad \mathfrak{p} \in \mathfrak{P}_{\mathfrak{n}}, \qquad d_{[n]} = \underbrace{(d, d, \dots, d)}_{n},$$
and $K_{\mathfrak{p}(1:n)}(d_{[n]})$ is the permutation matrix see (26).
2. Multilinearity.
   - For any constant matrices A and B
$$\begin{aligned}\underline{\operatorname{Cum}}_{n+1}\big(A\underline{Y}_1 + B\underline{Y}_2, \boldsymbol{X}_{(1:n)}\big) &= (A \otimes I_{d^n})\underline{\operatorname{Cum}}_{n+1}\big(\underline{Y}_1, \boldsymbol{X}_{(1:n)}\big)\\ &\quad + (B \otimes I_{d^n})\underline{\operatorname{Cum}}_{n+1}\big(\underline{Y}_2, \boldsymbol{X}_{(1:n)}\big),\end{aligned}$$
assuming that the appropriate matrix operations exist.
   - For any constant vectors $\boldsymbol{a}$ and $\boldsymbol{b}$
$$\begin{aligned}\underline{\operatorname{Cum}}_{n+1}\big(\boldsymbol{a} \otimes \underline{Y}_1 + \boldsymbol{b} \otimes \underline{Y}_2, \boldsymbol{X}_{(1:n)}\big) &= \boldsymbol{a} \otimes \underline{\operatorname{Cum}}_{n+1}\big(\underline{Y}_1, \boldsymbol{X}_{(1:n)}\big)\\ &\quad + \boldsymbol{b} \otimes \underline{\operatorname{Cum}}_{n+1}\big(\underline{Y}_2, \boldsymbol{X}_{(1:n)}\big),\end{aligned}$$
assuming that the appropriate vector addition exist.
3. Independence. If $\boldsymbol{X}_{(1:n)}$ is independent of $\boldsymbol{Y}_{(1:m)}$, where $n, m > 0$, then
$$\underline{\operatorname{Cum}}_{n+m}\big(\boldsymbol{X}_{(1:n)}, \boldsymbol{Y}_{(1:m)}\big) = 0.$$
In particular if the dimensions are same then
$$\underline{\operatorname{Cum}}_n\big(\boldsymbol{X}_{(1:n)} + \boldsymbol{Y}_{(1:n)}\big) = \underline{\operatorname{Cum}}_n\big(\boldsymbol{X}_{(1:n)}\big) + \underline{\operatorname{Cum}}_n\big(\boldsymbol{Y}_{(1:n)}\big).$$
4. Gaussianity. The random vector $\boldsymbol{X}_{(1:n)}$ is Gaussian if and only if for all subset $k_{(1:m)}$ of $(1:n)$
$$\underline{\operatorname{Cum}}_m(\boldsymbol{X}_{k_{(1:m)}}) = 0, \qquad m > 2.$$
5. Expression of the cumulant via moments. The expectation operator E defined for vector or matrices acting by element wise, for example $\mathsf{E}(X_1, X_2) = (\mathsf{E}\,X_1, \mathsf{E}\,X_2)$. Now the formula is
$$\begin{aligned}\underline{\operatorname{Cum}}_n(\boldsymbol{X}_{(1:n)}) &= \sum_{m=1}^{n}(-1)^{m-1}(m-1)!\\ &\qquad \times \sum_{\substack{\mathcal{L}\in\mathcal{P}_{(1:n)}\\ |\mathcal{L}|=m}} K^{-1}_{\mathfrak{p}(\mathcal{L})}(d_{(1:n)}) \prod\nolimits^{\otimes}_{j=1:m} \mathsf{E} \prod\nolimits^{\otimes}_{k\in\boldsymbol{b}_j} \underline{X}_k,\end{aligned} \tag{3}$$
where the second summation is taken over all possible ordered partition $\mathcal{L} \in \mathcal{P}_{(1:n)}$ with $|\mathcal{L}| = m$, see section A.1 for more details.

Example 2. Note that the permutations are depending on the ordered partitions therefore we use also the notation $\mathfrak{p}(\mathcal{L})$. Now we apply (3):
$$\begin{aligned}\underline{\operatorname{Cum}}_3(\boldsymbol{X}_{(1:3)}) &= \mathsf{E}(\underline{X}_1 \otimes \underline{X}_2 \otimes \underline{X}_3) - \mathsf{E}\,\underline{X}_1 \otimes \mathsf{E}(\underline{X}_2 \otimes \underline{X}_3)\\ &\quad - K^{-1}_{\mathfrak{p}_{2\to1}}(d_{1:3})\,\mathsf{E}\,\underline{X}_2 \otimes \mathsf{E}(\underline{X}_1 \otimes \underline{X}_3) - \mathsf{E}(\underline{X}_1 \otimes \underline{X}_2) \otimes \mathsf{E}\,\underline{X}_3\\ &\quad + 2\,\mathsf{E}\,\underline{X}_1 \otimes \mathsf{E}\,\underline{X}_2 \otimes \mathsf{E}\,\underline{X}_3\\ &= \mathsf{E}\prod\nolimits^{\otimes}_{i=1:3}(\underline{X}_i - \mathsf{E}\,\underline{X}_i).\end{aligned}$$

The third term of the right hand side may be replaced by the formula

$$K^{-1}_{\mathfrak{p}_{2\to 1}}(d_{1:3})\,\mathsf{E}\,\underline{X}_2 \otimes \mathsf{E}(\underline{X}_1 \otimes \underline{X}_3) = \mathsf{E}(\underline{X}_1 \otimes \mathsf{E}\,\underline{X}_2 \otimes \underline{X}_3).$$

The first three cumulants equal the central moments but this is not true for higher order cumulants. One might easily check this for the case of cumulants of order four.

6. Expression of the moment via cumulants. We consider the moment $\mathsf{E}\,\boldsymbol{X}^{\otimes \mathbf{1}_{[n]}}_{(1:n)}$ as the general case because the moment $\mathsf{E}\,\boldsymbol{Y}^{\otimes k_{(1:m)}}_{(1:m)}$ can be put into the form $\mathsf{E}\,\boldsymbol{X}^{\otimes \mathbf{1}_{[n]}}_{(1:n)}$, where

$$\boldsymbol{X}_{(1:n)} = \left(\underline{Y}_{1_{[k_1]}}, \dots, \underline{Y}_{m_{[k_m]}}\right) = (\underbrace{\underline{Y}_1, \dots, \underline{Y}_1}_{k_1}, \dots, \underbrace{\underline{Y}_m, \dots, \underline{Y}_m}_{k_m}),$$

i.e., the same elements in the product $\boldsymbol{Y}^{\otimes k_{(1:m)}}_{(1:m)}$ are treated as they were different.

$$\mathsf{E}\,\boldsymbol{X}^{\mathbf{1}_{[n]}}_{(1:n)} = \sum_{\mathcal{L}\in\mathcal{P}_{(1:n)}} K^{-1}_{\mathfrak{p}(\mathcal{L})}\left(d_{(1:n)}\right) \prod\nolimits^{\otimes}_{\boldsymbol{b}_j\in\mathcal{L}} \underline{\mathrm{Cum}}_n\left(\underline{X}_{\boldsymbol{b}_j}\right), \tag{4}$$

where the summation is over all ordered partitions $\mathcal{L} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_k\}$ of $(1:n)$.

Example 3. An example of the use of the formula (4) can be seen when $n = 3$,

$$\begin{aligned}\mathsf{E}\,\boldsymbol{X}^{\otimes \mathbf{1}_{[n]}}_{(1:3)} &= \underline{\mathrm{Cum}}_3\left(\underline{X}_{(1:3)}\right) + \underline{\mathrm{Cum}}_1(\underline{X}_1) \otimes \underline{\mathrm{Cum}}_2(\underline{X}_2, \underline{X}_3)\\ &\quad + K^{-1}_{\mathfrak{p}_{2,1}}(d_{1:3})\underline{\mathrm{Cum}}_1(\underline{X}_2) \otimes \mathrm{Cum}_2(\underline{X}_1, \underline{X}_3) + \underline{\mathrm{Cum}}_2(\underline{X}_1, \underline{X}_2) \otimes \underline{\mathrm{Cum}}_1(\underline{X}_3)\\ &\quad + \underline{\mathrm{Cum}}_1(\underline{X}_1) \otimes \underline{\mathrm{Cum}}_1(\underline{X}_2) \otimes \underline{\mathrm{Cum}}_1(\underline{X}_3),\end{aligned}$$

Now in particular

$$\begin{aligned}\mathsf{E}\,\underline{X}^{\otimes 3} &= \underline{\mathrm{Cum}}_3(\underline{X}, \underline{X}, \underline{X})\\ &\quad + \left(I + K^{-1}_{\mathfrak{p}_{2\leftrightarrow 1}}(d_{[3]}) + K^{-1}_{\mathfrak{p}_{2\leftrightarrow 1}}(d^2, d)\right)\underline{\mathrm{Cum}}_1(\underline{X}) \otimes \underline{\mathrm{Cum}}_2(\underline{X}, \underline{X}) + \underline{\mathrm{Cum}}_1(\underline{X})^{\otimes 3}.\end{aligned}$$

7. Expression of the cumulant of products via products of cumulants.Let $\boldsymbol{X}_{\mathcal{K}}$ denote the vector of entries taken by the partition $\mathcal{K}$, i.e., if $\mathcal{K} = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \dots, \boldsymbol{b}_m\}$, then $\boldsymbol{X}_{\mathcal{K}} = (\prod^{\otimes} \boldsymbol{X}_{\boldsymbol{b}_1}, \prod^{\otimes} \boldsymbol{X}_{\boldsymbol{b}_2}, \dots, \prod^{\otimes} \boldsymbol{X}_{\boldsymbol{b}_m})$. The order of the elements of the subsets $\boldsymbol{b} \in \mathcal{K}$ and the order of the subsets in $\mathcal{K}$ are fixed. Now the cumulant of the products can be expressed by the cumulants of the individual set of variables $\boldsymbol{X}_{\boldsymbol{b}} = (\underline{X}_j, j \in \boldsymbol{b})$, $\boldsymbol{b} \in \mathcal{L}$, such that $\mathcal{K} \cup \mathcal{L} = \mathcal{O}$, i.e., the partitions $\mathcal{L}$ and $\mathcal{K}^{\mathrm{II}}$ are indecomposable,

$$\begin{aligned}&\underline{\mathrm{Cum}}_k\left(\left(\prod\nolimits^{\otimes} \boldsymbol{X}_{\boldsymbol{b}_1}, \prod\nolimits^{\otimes} \boldsymbol{X}_{\boldsymbol{b}_2}, \dots, \prod\nolimits^{\otimes} \boldsymbol{X}_{\boldsymbol{b}_m}\right)\right)\\ &\qquad = \sum_{\mathcal{K}\cup\mathcal{L}=\mathcal{O}} K^{-1}_{\mathfrak{p}(\mathcal{L})}\left(d_{(1:n)}\right) \prod\nolimits^{\otimes}_{\boldsymbol{b}\in\mathcal{L}} \underline{\mathrm{Cum}}_{|\boldsymbol{b}|}(\boldsymbol{X}_{\boldsymbol{b}}),\end{aligned} \tag{5}$$

where $\boldsymbol{X}_{\boldsymbol{b}}$ denotes the set of vectors containing the items $\underline{X}_s$, $s \in \boldsymbol{b}$.

3. VECTOR HERMITE POLYNOMIALS OF SEVERAL VARIABLES

Classical Hermite polynomials $H_n(x)$ of degree n and with a positive leading coefficient were originally defined by the following conditions

$$\int_{-\infty}^{+\infty} H_n(x)H_m(x)\frac{1}{\sqrt{2\pi}}\exp\left(-x^2/2\right)\,dx = n!\,\delta_{\{n=m\}}, \qquad n,m=0,1,2,\dots,$$

see [17]. These polynomials form a closed orthogonal system in the Hilbert space

$$\mathfrak{L}^2 = \mathfrak{L}^2\left(\mathbf{R},\mathcal{B},\frac{1}{\sqrt{2\pi}}\exp\left(-x^2/2\right)\,dx\right),$$

see [23]. The generalization of this fact is the following. Let $\mathcal{X}=\{X_t, t\in T\}$ be a Gaussian system, put $\mathfrak{L}^2(\mathcal{X})$ as the Hilbert space of all random variables depending on $\mathcal{X}$. It is known that there exist a closed orthogonal system, the elements of which are polynomials in X_t, in $\mathfrak{L}^2(\mathcal{X})$, see Ibragimov and Rosanov [9, p. 27].

Consider a Gaussian system $\mathcal{X}=\left\{\underline{X}_t\in\mathbf{R}^{d_t}, t=1,2,\dots\right\}$ with $\mathsf{E}\,\underline{X}_t=0$, $t=1,2,\dots,$ and put $C_{j,k}=\underline{\underline{\mathrm{Cov}}}(\underline{X}_j,\underline{X}_k)=\mathsf{E}\,\underline{X}_j\underline{X}_k'$, i.e., $C_{j,k}$ is the covariance matrix of $\underline{X}_j$ and $\underline{X}_k$. Put

$$\psi\left(\boldsymbol{X}_{(1:n)},\boldsymbol{a}_{(1:n)}\right)=\exp\left(\sum_{k=1}^{n}\underline{a}_k'\underline{X}_k-\frac{1}{2}\sum_{k,j=1}^{n}\underline{a}_k'C_{k,j}\underline{a}_j\right).$$

Definition 4. $\underline{H}_0=1$, for $m\ge 1$ the mth order Hermite polynomial of the set $\boldsymbol{X}_{(1:n)}=(\underline{X}_1,\underline{X}_2,\dots,\underline{X}_n)$ of Gaussian system is defined by the equation

$$\underline{H}_m\left(\boldsymbol{X}_{(1:m)}\right)=D^{\otimes m}_{\boldsymbol{a}_{(1:m)}}\psi\left(\boldsymbol{X}_{(1:n)},\boldsymbol{a}_{(1:n)}\right)\Big|_{\boldsymbol{a}_{(1:n)}=0}. \tag{6}$$

In particular we have $\underline{H}_1(\underline{X}_k)=\underline{X}_k$, $\underline{H}_2(\underline{X}_k,\underline{X}_j)=\underline{X}_k\otimes\underline{X}_j-\mathrm{Vec}(C_{j,k})$, $\dots$. The vector $\underline{H}_m$ is with dimension $\prod d_{(1:m)}$ and we shall see that all entries are Hermite polynomials of order m. $\underline{H}_m$ is not symmetric as a function of vectors but the entries are symmetric as the function of scalars. The generating function $\psi(\boldsymbol{X}_{(1:n)},\boldsymbol{a}_{(1:n)})$ can be rewritten

$$\begin{aligned}\psi\left(\boldsymbol{X}_{(1:n)},\boldsymbol{a}_{(1:n)}\right)&=\exp\left(\sum_{k=1}^{n}\underline{a}_k'\underline{X}_k-\frac{1}{2}\sum_{k,j=1}^{n}\underline{a}_k'C_{k,j}\underline{a}_j\right)\\&=\exp\left(\sum_{k=1}^{n}\underline{a}_k'\underline{X}_k\right)\mathsf{E}\exp\left(i\sum_{k=1}^{n}\underline{a}_k'\underline{X}_k\right)\\&=\mathsf{E}\left[\exp\left(\sum_{k=1}^{n}\underline{a}_k'\left(\underline{X}_k+i\underline{Y}_k\right)\right)\Bigg|\,\boldsymbol{X}_{(1:n)}\right],\end{aligned}$$

where $\underline{\boldsymbol{Y}}_{(1:n)}$ is an independent copy of $\boldsymbol{X}_{(1:n)}$.

Lemma 5. *The above conditional expectation implies*

1. *The mth order Hermite polynomial of $\boldsymbol{X}_{(1:n)}$ has the representation*

$$\begin{aligned}\underline{H}_m(\boldsymbol{X}_{(1:m)})&=\mathsf{E}\left[\prod\nolimits_{k=1:m}^{\otimes}(\underline{X}_k+i\underline{Y}_k)\Big|\,\boldsymbol{X}_{(1:n)}\right]\\&=\mathsf{E}\prod\nolimits_{k=1:m}^{\otimes}(\underline{x}_k+i\underline{X}_k)\Big|_{\boldsymbol{X}_{(1:n)}=\boldsymbol{X}_{(1:n)}},\end{aligned}\tag{7}$$

where $\underline{\boldsymbol{Y}}_{(1:n)}$ is an independent copy of $\boldsymbol{X}_{(1:n)}$.

2. *If* $\boldsymbol{X}_{(1:k)}$ *and* $\boldsymbol{X}_{(k+1:n)}$ *are statistically independent then*

$$\underline{H}_n\big(\boldsymbol{X}_{(1:n)}\big) = \underline{H}_k\big(\boldsymbol{X}_{(1:k)}\big) \otimes \underline{H}_{n-k}\big(\boldsymbol{X}_{(k+1:n)}\big).$$

3. *It is not symmetric in general, instead it fulfils the equation*

$$\underline{H}_n\big(\boldsymbol{X}_{(1:n)}\big) = K^{-1}_{\mathfrak{p}(1:n)}\big(d_{(1:n)}\big)\underline{H}_n\big(\boldsymbol{X}_{\mathfrak{p}(1:n)}\big), \tag{8}$$

for any permutation $\mathfrak{p} \in \mathfrak{P}_{\mathfrak{n}}$.

Proposition 6. *The following recursion rule defines the Hermite polynomials*

- $\underline{H}_0 = 1$, $\underline{H}_1(\underline{X}) = \underline{X}$,
- *If* $n > 1$

$$\begin{aligned}\underline{H}_n\big(\boldsymbol{X}_{(1:n)}\big) &= \underline{H}_{n-1}\big(\boldsymbol{X}_{(1:n-1)}\big) \otimes \underline{X}_n \\ &\quad - \sum_{j=1}^{n-1} K^{-1}_{\mathfrak{p}_{(n,j)\to(1,2)}}\big(d_{(1:n)}\big) \\ &\qquad \times \big[\underline{\mathrm{Cum}}_2(\underline{X}_n, \underline{X}_j) \otimes \underline{H}_{n-2}\big(\boldsymbol{X}_{(1:j-1,j+1:n-1)}\big)\big],\end{aligned} \tag{9}$$

where the permutation $\mathfrak{p}_{(n,j)\to(1,2)}$ *is putting the* n*th element to the first and* j*th to the second place keep the others being unchanged, more precisely*

$$\begin{aligned}K^{-1}_{\mathfrak{p}_{(n,j)\to(1,2)}}\big(d_{(1:n)}\big) &= \big(K_{\mathfrak{p}_{j+1\to 2}}\big(d_n, d_{(1:n-1)}\big)K_{\mathfrak{p}_{n\to 1}}\big(d_{(1:n)}\big)\big)^{-1} \\ &= K^{-1}_{\mathfrak{p}_{n\to 1}}\big(d_{(1:n)}\big)K^{-1}_{\mathfrak{p}_{j+1\to 2}}\big(d_n, d_{(1:n-1)}\big).\end{aligned}$$

If $j = 1$, *then the variable* $\underline{X}_1$ *is missing from* $\underline{H}_{n-2}$, *i.e., it is* $\underline{H}_{n-2}(\boldsymbol{X}_{(2:n-1)})$.

Proof. Indeed, it is easy to see that

$$D^{\otimes m-1}_{\underline{\boldsymbol{a}}_{(1:m-1)}}\psi\big(\boldsymbol{X}_{(1:n)}, \boldsymbol{a}_{(1:n)}\big) = \psi\big(\boldsymbol{X}_{(1:n)}, \boldsymbol{a}_{(1:n)}\big)\boldsymbol{L}_{m-1}\big(\boldsymbol{X}_{(1:m-1)}, \boldsymbol{a}_{(1:n)}\big),$$

where the vector polynomial $\boldsymbol{L}_{m-1}(\boldsymbol{X}_{(1:m-1)}, \boldsymbol{a}_{(1:n)})$ is with dimension $\prod d_{(1:m-1)}$ and by the formula (6)

$$\boldsymbol{L}_{m-1}\big(\boldsymbol{X}_{(1:m-1)}, \boldsymbol{a}_{(1:n)}\big)\big|_{\boldsymbol{a}_{(1:n)}=0} = \underline{H}_{m-1}\big(\boldsymbol{X}_{(1:m-1)}\big)$$

one can express it as

$$\begin{aligned}\boldsymbol{L}_{m-1}\big(\boldsymbol{X}_{(1:m-1)}, \boldsymbol{a}_{(1:n)}\big) &= \psi^{-1}\big(\boldsymbol{X}_{(1:n)}, \boldsymbol{a}_{(1:n)}\big)D^{\otimes m-1}_{\boldsymbol{a}_{(1:m-1)}}\psi\big(\boldsymbol{X}_{(1:n)}, \boldsymbol{a}_{(1:n)}\big) \\ &= \psi^{-1}\big(\boldsymbol{X}_{(1:m-1)}, \boldsymbol{a}_{(1:m-1)}\big)\exp\left(\frac{1}{2}\sum_{\substack{j=1:m-1\\k=m:n}}\big[\underline{a}'_k C_{k,j}\underline{a}_j + \underline{a}'_j C_{j,k}\underline{a}_k\big]\right) \\ &\quad \times D^{\otimes m-1}_{\boldsymbol{a}_{(1:m-1)}}\left[\psi\big(\boldsymbol{X}_{(1:m-1)}, \boldsymbol{a}_{(1:m-1)}\big)\exp\left(-\frac{1}{2}\sum_{\substack{j=1:m-1\\k=m:n}}\big[\underline{a}'_k C_{k,j}\underline{a}_j + \underline{a}'_j C_{j,k}\underline{a}_k\big]\right)\right].\end{aligned}$$

Differentiate it and obtain

$$\begin{aligned}
&D^{\otimes}_{\underline{a}_m}\boldsymbol{L}_{m-1}\bigl(\boldsymbol{X}_{(1:m-1)},\boldsymbol{a}_{(1:n)}\bigr)\Big|_{\boldsymbol{a}_{(1:n)}=0}\\
&\quad= K^{-1}_{\mathfrak{p}_{m\to 1}}(d_{1:m})\, D^{\otimes m-1}_{\boldsymbol{a}_{(1:m-1)}}\left[\psi\bigl(\boldsymbol{X}_{(1:m-1)},\boldsymbol{a}_{(1:m-1)}\bigr)\left(-\sum_{j=1:m-1} C_{m,j}\underline{a}_j\right)\right]\Bigg|_{\boldsymbol{a}_{(1:n)}=0}\\
&\quad= -K^{-1}_{\mathfrak{p}_{1\to m}}(d_{1:m})\\
&\qquad\times\sum_{j=1:m-1} K^{-1}_{\mathfrak{p}_{j+1\to 2}}(d_m,d_{1:m-1})\bigl[\operatorname{Vec}(C_{j,m})\otimes\underline{H}_{m-2}\bigl(\boldsymbol{X}_{(1:j-1,j+1:m-1)}\bigr)\bigr]
\end{aligned}$$

Now by the definition

$$\begin{aligned}
\underline{H}_m\bigl(\boldsymbol{X}_{(1:m)}\bigr) &= D^{\otimes m}_{\boldsymbol{a}_{(1:m)}}\psi\bigl(\boldsymbol{X}_{(1:n)},\boldsymbol{a}_{(1:n)}\bigr)\Big|_{\boldsymbol{a}_{(1:n)}=0}\\
&= \bigl(D_{\underline{a}_m}D^{\otimes m-1}_{\boldsymbol{a}_{(1:m-1)}}\bigr)\psi\bigl(\boldsymbol{X}_{(1:n)},\boldsymbol{a}_{(1:n)}\bigr)\Big|_{\boldsymbol{a}_{(1:n)}=0}\\
&= \bigl(D_{\underline{a}_m}\psi\bigl(\boldsymbol{X}_{(1:n)},\boldsymbol{a}_{(1:n)}\bigr)\boldsymbol{L}_{m-1}\bigl(\boldsymbol{X}_{(1:m-1)},\boldsymbol{a}_{(1:n)}\bigr)\bigr)\Big|_{\boldsymbol{a}_{(1:n)}=0}\\
&= \underline{H}_{m-1}\bigl(\boldsymbol{X}_{(1:m-1)}\bigr)\otimes\underline{X}_m - K^{-1}_{\mathfrak{p}_{1\to m}}(d_{1:m})\\
&\quad\times\sum_{j=1:m-1} K^{-1}_{\mathfrak{p}_{j+1\to 2}}(d_m,d_{1:m-1})\bigl[\operatorname{Vec}(C_{j,m})\otimes\underline{H}_{m-2}\bigl(\boldsymbol{X}_{(1:j-1,j+1:m-1)}\bigr)\bigr].
\end{aligned}$$

Remember that $\operatorname{Vec}(C_{j,n}) = \underline{\operatorname{Cum}}_2(\underline{X}_n,\underline{X}_j) = \mathsf{E}\,\underline{X}_n\otimes\underline{X}_j$, hence the result of the Lemma. ☐

Note that the Hermite polynomials depend on the covariance structure of the underlying Gaussian system. The list of the first 5 polynomials follows:

$$\begin{aligned}
&\underline{H}_0 = 1,\qquad \underline{H}_1(\underline{X}_1) = \underline{X}_1,\\
\underline{H}_2(\boldsymbol{X}_{(1:2)}) &= \underline{X}_1\otimes\underline{X}_2 - K^{-1}_{\mathfrak{p}_{2\to 1}}(d_{1:2})\operatorname{Vec}C_{1,2} = \underline{X}_1\otimes\underline{X}_2 - \operatorname{Vec}C_{2,1},\\
\underline{H}_3(\boldsymbol{X}_{(1:3)}) &= \underline{X}_1\otimes\underline{X}_2\otimes\underline{X}_3 - \operatorname{Vec}(C_{2,1})\otimes\underline{X}_3 - K^{-1}_{\mathfrak{p}_{3\to 1}}(d_{1:3})[\operatorname{Vec}C_{1,3}\otimes\underline{X}_2]\\
&\quad - \underline{X}_1\otimes\operatorname{Vec}C_{3,2},\\
\underline{H}_4(\boldsymbol{X}_{(1:4)}) &= \underline{X}_1\otimes\underline{X}_2\otimes\underline{X}_3\otimes\underline{X}_4 - \operatorname{Vec}C_{2,1}\otimes\underline{X}_3\otimes\underline{X}_4\\
&\quad - K^{-1}_{\mathfrak{p}_{3\to 1}}(d_{1:4})[\operatorname{Vec}C_{1,3}\otimes\underline{X}_2\otimes\underline{X}_4] - \underline{X}_1\otimes\operatorname{Vec}(C_{3,2})\otimes\underline{X}_4\\
&\quad - K^{-1}_{\mathfrak{p}_{4\to 1}}(d_{1:4})[\operatorname{Vec}C_{1,4}\otimes\underline{X}_2\otimes\underline{X}_3 - \operatorname{Vec}C_{1,4}\otimes\operatorname{Vec}C_{3,2}]\\
&\quad - K^{-1}_{\mathfrak{p}_{(4,2)\to(1,2)}}(d_{1:4})[\operatorname{Vec}C_{2,4}\otimes\underline{X}_1\otimes\underline{X}_3 - \operatorname{Vec}C_{2,4}\otimes\operatorname{Vec}C_{3,1}]\\
&\quad - \underline{X}_1\otimes\underline{X}_2\otimes\operatorname{Vec}C_{4,3} + \operatorname{Vec}C_{2,1}\otimes\operatorname{Vec}C_{4,3}.
\end{aligned}$$

We note that $K^{-1}_{\mathfrak{p}_{(4,3)\to(1,2)}}(d_{1:4})\operatorname{Vec}C_{3,4}\otimes\underline{X}_1\otimes\underline{X}_2 = \underline{X}_1\otimes\underline{X}_2\otimes\operatorname{Vec}C_{4,3}$.

Proposition 7. *Further properties of Hermite polynomials:*

1. $D^{\otimes}_{\underline{X}_j}\underline{H}_n(\boldsymbol{X}_{(1:n)}) = (K^{-1}_{\mathfrak{p}_{j\to n}}(d_{(1:n)})\otimes I_{d_j})\operatorname{Vec}(\underline{H}'_{n-1}(\boldsymbol{X}_{(1:j-1,j+1:n)})\otimes I_{d_j})$. *In particular, suppose that all* $\underline{X}_j = \underline{X}$, *then*

$$D^{\otimes}_{\underline{X}}\underline{H}_n(\boldsymbol{X}_{(1:n)}) = \left[\sum_{j=1}^{n} K^{-1}_{\mathfrak{p}_{j\to n}}(d_{[n]})\otimes I_d\right]\operatorname{Vec}\left[\underline{H}'_{n-1}(\boldsymbol{X}_{(1:n-1)})\otimes I_d\right].$$

2. *Multilinearity. If* $\left\{\underline{Y},\underline{Z},\boldsymbol{X}_{(1:n)}\right\}$ *is a Gaussian system and*

- *A and B are appropriate real matrices, then*
$$\underline{H}_{n+1}\left(A\underline{Y}+B\underline{Z},\boldsymbol{X}_{(1:n)}\right)=(A\otimes I_{d^n})\underline{H}_{n+1}\left(\underline{Y},\boldsymbol{X}_{(1:n)}\right)+(B\otimes I_{d^n})\underline{H}_{n+1}\left(\underline{Z},\boldsymbol{X}_{(1:n)}\right).$$
- *$\boldsymbol{a}$ and $\boldsymbol{b}$ are real vectors:*
$$\underline{H}_{n+1}\left(\boldsymbol{a}\otimes\underline{Y}+\boldsymbol{b}\otimes\underline{Z},\boldsymbol{X}_{(1:n)}\right)=\boldsymbol{a}\otimes\underline{H}_{n+1}\left(\underline{Y},\boldsymbol{X}_{(1:n)}\right)+\boldsymbol{b}\otimes\underline{H}_{n+1}\left(\underline{Z},\boldsymbol{X}_{(1:n)}\right),$$
- *If $\boldsymbol{Y}_{(1:n)}=(A_1\underline{X}_1,A_2\underline{X}_2,\dots,A_n\underline{X}_n)$, then*
$$\underline{H}_n\left(\underline{\boldsymbol{Y}}_{(1:n)}\right)=\left(\prod\nolimits_{k=1:n}^{\otimes}A_k\right)\underline{H}_n\left(\boldsymbol{X}_{(1:n)}\right).$$

Proof.

1. It follows from the formulae (9) and (8) that
$$\begin{aligned}D_{\underline{X}_j}^{\otimes}\underline{H}_n\left(\boldsymbol{X}_{(1:n)}\right)&=\operatorname{Vec}\left(\left[K_{\mathfrak{p}_{j\to n}}^{-1}\left(d_{(1:n)}\right)\left(\underline{H}_{n-1}(\boldsymbol{X}_{(1:j-1,j+1:n)})\otimes\underline{X}_j\right)\right]\frac{\partial}{\partial\underline{X}_j'}\right)'\\&\quad\times\operatorname{Vec}\left(K_{\mathfrak{p}_{j\to n}}^{-1}\left(d_{(1:n)}\right)\left[\underline{H}_{n-1}\left(\boldsymbol{X}_{(1:j-1,j+1:n)}\right)\otimes I_{d_j}\right]\right)'\\&=\left(K_{\mathfrak{p}_{j\to n}}^{-1}(d_{(1:n)})\otimes I_{d_j}\right)\operatorname{Vec}\left(\underline{H}_{n-1}'\left(\boldsymbol{X}_{(1:j-1,j+1:n)}\right)\otimes I_{d_j}\right).\end{aligned}$$
2. These statements follow from the property (7). ☐

Example 8. For instance as $K_{1\to 2}^{-1}(1,d_k)=I$ we obtain
$$\underline{H}_2\left(\boldsymbol{a}'\underline{X}_k,\boldsymbol{b}'\underline{X}_j\right)=(\boldsymbol{a}'\otimes\boldsymbol{b}')\underline{H}_2\left(\underline{X}_k,\underline{X}_j\right).$$
Now
$$(\boldsymbol{a}'\otimes\boldsymbol{b}')\underline{H}_2\left(\underline{X}_k,\underline{X}_j\right)=(\boldsymbol{a}'\otimes\boldsymbol{b}')\left[\underline{X}_k\otimes\underline{X}_j-\operatorname{Vec}(\underline{\underline{\operatorname{Cov}}}(\underline{X}_j,\underline{X}_k))\right]=H_2\left(\boldsymbol{a}'\underline{X}_k,\boldsymbol{b}'\underline{X}_j\right).$$
Therefore
$$\underline{H}_2\left(\boldsymbol{a}'\underline{X}_k,\boldsymbol{b}'\underline{X}_j\right)=H_2\left(\boldsymbol{a}'\underline{X}_k,\boldsymbol{b}'\underline{X}_j\right),$$
i.e., the definition of vector $\underline{H}_2$ is compatible to the definition of H_2.

The (7) implies also that
$$\underline{H}_n\left(\boldsymbol{X}_{(1:j-1)},\boldsymbol{a}\otimes\underline{X}_j,\boldsymbol{X}_{(j+1,n)}\right)=K_{\mathfrak{p}_{j\to 1}}^{-1}\left(d_{1:j-1},m,d_{j:n}\right)\left[\boldsymbol{a}\otimes\underline{H}_n\left(\boldsymbol{X}_{(1:n)}\right)\right],$$
where $\boldsymbol{a}\in\mathbf{R}^m$.

Proposition 9. *Suppose that $\mathcal{X}=\{\underline{X}_t,t\in T\}$ is a Gaussian system. If*
$$\boldsymbol{X}_{t_{(1:n)}}=\left(\underline{X}_{t_1},\underline{X}_{t_2},\dots,\underline{X}_{t_n}\right)$$
is a finite subset of $\mathcal{X}$, then
$$\mathsf{E}[\underline{H}_n(\boldsymbol{X}_{t_{(1:n)}})|\,\mathcal{X}]=\underline{H}_n\left(\mathsf{E}(\underline{X}_{t_1}|\,\mathcal{X}),\mathsf{E}(\underline{X}_{t_2}|\,\mathcal{X}),\dots,\mathsf{E}(\underline{X}_{t_n}|\,\mathcal{X})\right).$$

Proof. The proof follows from the orthogonality of Hermite polynomials of different orders and the second order cumulant of same orders, see (10). ☐

As $\underline{X}_t=\underline{X}$, $t=1,2,\dots$, is a Gaussian system therefore the above definitions are generalizations of those given in [24]. The Hermite polynomials depend strongly on the covariance structure of the Gaussian variables under consideration therefore we shall often put Gaussian variables as the variables of the polynomials to make this fact clear.

4. Moments and cumulants for Gaussian systems

4.1. Moments of Gaussian systems. Let us consider the higher order moments for a Gaussian system $\mathcal{X} = \{\underline{X}_t, t \in T\}$ with $\mathsf{E}\,\underline{X}_t = 0$ and covariance $C_{s,t} = \underline{\underline{\mathrm{Cov}}}(\underline{X}_s, \underline{X}_t)$, where $T \subseteq \mathbf{Z} = \{0, \pm 1, \pm 2, \dots\}$. We have seen that if n is even, then

$$\mathsf{E}(\underline{X}_1 \otimes \underline{X}_2 \otimes \cdots \otimes \underline{X}_n) = \sum_{\mathcal{K}^{\mathrm{II}} \in \mathcal{P}^{\mathrm{II}}_{(1:n)}} K^{-1}_{\mathfrak{p}(\mathcal{K}^{\mathrm{II}})}\left(d_{(1:n)}\right) \prod\nolimits^{\otimes}_{(j,k) \in \mathcal{K}^{\mathrm{II}}} \operatorname{Vec} C'_{jk},$$

where the summation is taken over all the possible partitions $\mathcal{K}^{\mathrm{II}}$ of $\mathcal{P}^{\mathrm{II}}_{(1:n)}$, the partitions into pairs of the set $(1 : n) = (1, 2, \dots, n)$. The first order moments of the Hermite polynomials are zero by the construction, the second order ones are

$$\begin{aligned} &\mathsf{E}\big[\underline{H}_n(\boldsymbol{X}_{t_{(1:n)}}) \otimes \underline{H}_m(\boldsymbol{X}_{s_{(1:m)}})\big] \\ &\quad \times = \delta_{\{n=m\}} \sum_{n!} K^{-1}_{\mathfrak{q}(1:n,\mathfrak{p}(1:n))}\big(d_{t_{(1:n)}}, d_{s_{(1:n)}}\big) \prod\nolimits^{\otimes}_{j=1:n} \operatorname{Vec} C'_{t_j, s_{\mathfrak{p}(j)}}, \end{aligned} \tag{10}$$

where $t_{(1:n)} = (t_1, t_2, \dots, t_n)$, $s_{(1:m)} = (s_1, s_1, \dots, s_m)$ and the summation is taken for all the permutations$\mathfrak{q}(1 : n, \mathfrak{p}(1 : n)) \in \mathcal{P}_{(1:2n)}$ defined by the following way. For each $\mathfrak{p}(1 : n) \in \mathcal{P}_{(1:n)}$ fixed, $\mathfrak{q}(2k + 1) = k$, $k = 1, \dots, n$, and $\mathfrak{q}(2k) = n + \mathfrak{p}(k)$, $k = 1, \dots, n$. For example if $n = 3$ and $\mathfrak{p} = (3, 2, 1)$ then $\mathfrak{q}(1 : 3, \mathfrak{p}) = (1, 6, 2, 5, 3, 4)$, i.e., for the indices t_j, $s_{\mathfrak{p}(j)}$, $j = 1, 2, 3$, stand t_1, s_3, t_2, $s_2 t_3$, s_1. The summation in (10) is taken for the indices $(s_{\mathfrak{p}(1)}, s_{\mathfrak{p}(2)} \dots, s_{\mathfrak{p}(n)})$ while the order of $(t_1, t_2, \dots, t_n)$ is fixed. In particular

$$\mathsf{E}\big[\underline{H}_n(\underline{X}_s) \otimes \underline{H}_n(\underline{X}_t)\big] = \left[\sum_{n!} K^{-1}_{\mathfrak{q}(1:n,\mathfrak{p}(1:n))}\left(d_{s_{[n]}}, d_{t_{[n]}}\right)\right] \operatorname{Vec}(C'_{t,s})^{\otimes n},$$

where $\underline{H}_n(\underline{X}_s) = \underline{H}_n(\underbrace{\underline{X}_s, \dots, \underline{X}_s}_{n})$. Next take a partition $\mathcal{L} = (L_1, L_2, \dots, L_p)$ of $(1 : n)$ with $|L_j| = n_j$, i.e., the number of indices in the set L_j is n_j. Then

$$\mathsf{E} \prod\nolimits^{\otimes}_{j=1:p} \underline{H}_{n_j}(\boldsymbol{X}_{L_j}) = \sum_{\left\{\mathcal{K}^{\mathrm{II}} \,\middle|\, \substack{(\mathcal{L},\mathcal{K}^{\mathrm{II}}) \\ \text{without loop}}\right\}} K^{-1}_{\mathfrak{p}(\mathcal{K}^{\mathrm{II}})}(d_{L_{1:p}}) \prod\nolimits^{\otimes}_{(t_j,t_k) \in \mathcal{K}^{\mathrm{II}}} \operatorname{Vec} C'_{t_j,t_k},$$

where the summation is taken over all diagrams $(\mathcal{L}, \mathcal{K}^{\mathrm{II}})$ without loops, see [15], [24] for more details. If the set $\left\{\mathcal{K}^{\mathrm{II}} \,\middle|\, \substack{(\mathcal{L},\mathcal{K}^{\mathrm{II}}) \\ \text{without loop}}\right\}$ is empty, the expectation is zero. In particular if $p = 2$, i.e., $\mathcal{L} = (L_1, L_2)$, then the diagram $(\mathcal{L}, \mathcal{K}^{\mathrm{II}})$ without loop corresponds to a partition $\mathcal{K}^{\mathrm{II}}$ containing pairs (l_1, l_2) such that $l_1 \in L_1$ and $l_2 \in L_2$. One can reach all such types of partitions by fixing the first entries and permuting the second ones.

4.2. Cumulants for Hermite polynomials. We have seen in (5) that in general

$$\underline{\mathrm{Cum}}_p(\underline{X}_{\mathcal{L}}) = \sum_{\mathcal{L} \cup \mathcal{K} = \mathcal{O}} K^{-1}_{\mathfrak{p}(\mathcal{K})}(d_{\mathcal{K}}) \prod\nolimits^{\otimes}_{\boldsymbol{b} \in \mathcal{K}} \underline{\mathrm{Cum}}_{|\boldsymbol{b}|}(\underline{X}_{\boldsymbol{b}}). \tag{11}$$

Now because of the Gaussianity of the system all the cumulants are zero on the right hand side of (11) except for the second order ones. Therefore the summation is taken only for $\mathcal{K} \in \mathcal{P}^{\mathrm{II}}_{(1:n)}$. The assumption $\mathcal{L} \cup \mathcal{K}^{\mathrm{II}} = \mathcal{O}$ is equivalent to the assumption that the graph $(\mathcal{L}, \mathcal{K}^{\mathrm{II}})$ is closed. We have thus the formula

$$\underline{\mathrm{Cum}}_p(\underline{X}_{\mathcal{L}}) = \sum_{\left\{\mathcal{K}^{\mathrm{II}} \,\middle|\, \substack{(\mathcal{L},\mathcal{K}^{\mathrm{II}}) \\ \text{closed}}\right\}} K^{-1}_{\mathfrak{p}(\mathcal{K}^{\mathrm{II}})}(d_{\mathcal{K}^{\mathrm{II}}}) \prod\nolimits^{\otimes}_{(t_i,t_j) \in \mathcal{K}^{\mathrm{II}}} \operatorname{Vec} C'_{t_i,t_j}. \tag{12}$$

Take again a partition $\mathcal{L} = (\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_p)$ of $(1:n)$ such that $|\boldsymbol{b}_j| = n_j$, then

$$\begin{aligned}\underline{\mathrm{Cum}}_p\left(\underline{H}_{n_1}(\underline{X}_{\boldsymbol{b}_1}), \underline{H}_{n_2}(\underline{X}_{\boldsymbol{b}_2}), \ldots, \underline{H}_{n_p}(\underline{X}_{\boldsymbol{b}_p})\right) \\ = \sum_{\left\{\mathcal{K}^{\mathrm{II}} \middle| \substack{(\mathcal{L},\mathcal{K}^{\mathrm{II}}) \text{ closed,} \\ \text{without loop}}\right\}} K^{-1}_{\mathfrak{p}(\mathcal{K}^{\mathrm{II}})}(d_{\boldsymbol{b}_{1:p}}) \prod\nolimits^{\otimes}_{(t_k,t_j)\in\mathcal{K}^{\mathrm{II}}} \operatorname{Vec} C_{t_j t_k},\end{aligned} \tag{13}$$

where the summation is over all closed diagrams $(\mathcal{L}, \mathcal{K}^{\mathrm{II}})$ without loops, see [15] for more details.

Suppose that all the variables below in the examples of this subsection are jointly Gaussian. The following equations will be useful for calculating higher order spectra for particular processes.

Example 10. For $k = 2$

$$\underline{\mathrm{Cum}}_2(\underline{H}_2(\underline{X}), \underline{H}_2(\underline{X})) = \mathsf{E}\,\underline{H}_2[\underline{X}]^{\otimes 2} = \left(K^{-1}_{\mathfrak{p}_{2\to 3}}\left(d_{[4]}\right) + K^{-1}_{\mathfrak{p}_{2\to 4}}\left(d_{[4]}\right)\right)(\operatorname{Vec} C)^{\otimes 2}.$$

For $k = 3$

$$\underline{\mathrm{Cum}}_3(\underline{H}_2(\underline{X}), \underline{H}_2(\underline{X}), \underline{H}_2(\underline{X})) = \mathsf{E}\,\underline{H}_2[\underline{X}]^{\otimes 3} = \left(\sum_{j=1}^{8} K^{-1}_{\mathfrak{p}_j}(d_{[6]})\right)(\operatorname{Vec} C)^{\otimes 3},$$

where $\mathfrak{p}_1 = (2,3,4,5,6,1)$, $\mathfrak{p}_2 = (2,3,4,5,6,1)$, $\mathfrak{p}_3 = (2,4,3,5,6,1)$, $\mathfrak{p}_4 = (2,4,3,6,5,1)$, $\mathfrak{p}_5 = (2,5,3,6,4,1)$, $\mathfrak{p}_6 = (2,5,4,6,3,1)$, $\mathfrak{p}_7 = (2,6,3,5,4,1)$, $\mathfrak{p}_8 = (2,6,4,5,3,1)$.

5. STATIONARY PROCESSES AND SPECTRA

A vector valued stochastic process $\underline{X}_t$, $t \in \mathbf{Z}$, is called stationary if the expectation $\mathsf{E}\,\underline{X}_t$ and the matrix covariance function

$$\underline{\underline{\mathrm{Cov}}}(\underline{X}_{t+s}, \underline{X}_t) = \mathsf{E}(\underline{X}_{t+s} - \mathsf{E}\,\underline{X}_{t+s})(\underline{X}_t - \mathsf{E}\,\underline{X}_t)'$$

are invariant under time shift, i.e., $\mathsf{E}\,\underline{X}_t = \mathsf{E}\,\underline{X}_0$, $\underline{\underline{\mathrm{Cov}}}(\underline{X}_{t+s}, \underline{X}_t) = \underline{\underline{\mathrm{Cov}}}(\underline{X}_s, \underline{X}_0)$. Let a stationary stochastic process $\underline{X}_t$, with $\mathsf{E}\,\underline{X}_t = 0$ and $\underline{\underline{\mathrm{Cov}}}(\underline{X}_t, \underline{X}_{t+s}) = \underline{\underline{C}}(s)$, be given. It is well known fact that one has the representation of the covariance function as

$$\underline{\underline{C}}(s) = \int_0^1 e^{i2\pi s\omega}\, \underline{\underline{\Phi}}(d\omega),$$

and the process $\underline{X}_t$ has Cramér representation as

$$\underline{X}_t = \int_0^1 e^{i2\pi t\omega}\, \underline{W}(d\omega),$$

for some matrix spectral measure $\underline{\underline{\Phi}}(d\omega)$ and vector stochastic spectral measure $\underline{W}(d\omega)$. These spectral measures are connected by the relationship

$$\mathsf{E}\,\underline{W}(d\omega)\,\underline{W}^*(d\omega) = \underline{\underline{\Phi}}(d\omega),$$

where $*$denotes the transpose and conjugate, i.e. $\underline{W}^*(d\omega) = \underline{W}'(-d\omega)$. The spectral measure $\underline{\underline{\Phi}}(d\omega)$ is defined on the Borel sets of the real line. It is periodic, $\underline{\underline{\Phi}}(d\omega + 1) = \underline{\underline{\Phi}}(d\omega)$, and symmetric about zero, $\underline{\underline{\Phi}}(-d\omega) = \underline{\underline{\Phi}}'(d\omega)$. If $\underline{X}_t$ is Gaussian, $\underline{W}(d\omega)$ is complex Gaussian, see [18], [2] for details.

5.1. Kronecker spectra. Suppose we have a stationary Gaussian stochastic process which is stationary and has all the higher order moments; then a large class of processes transformed from this process retain these properties. If a strictly stationary process$\underline{X}_t$ has third order moments, then not only the covariances $\underline{\mathrm{Cum}}_2(\underline{X}_{t+s}, \underline{X}_t) = \underline{c}(s)$ are invariant with respect to the time-shift but the third order cumulants

$$\underline{\mathrm{Cum}}_3(\underline{X}_{t+r}, \underline{X}_{t+s}, \underline{X}_t) = \underline{c}_3(r, s)$$

as well.

Definition 11. A stationary process $\underline{X}_t$ will be called stationary of kth order if all the cumulants exist up to kth order and are invariant with respect to the time shift, i.e., for all $t \in T, m = 1, 2, \dots k$, and $t_{(1:m)}$, $t_j \in T$,

$$\underline{\mathrm{Cum}}_m \left(\boldsymbol{X}_{t\mathbf{1}_{[m]}+t_{(1:m)}}\right) = \underline{\mathrm{Cum}}_m \left(\boldsymbol{X}_{t_{(1:m)}}\right).$$

For a stationary process of kth order the cumulants of mth ($m \leq k$) order depend on $m-1$ variables only

$$\underline{\mathrm{Cum}}_m \left(\boldsymbol{X}_{t_{(1:m)}}\right) = \underline{\mathrm{Cum}}_m \left(\underline{X}_0, \boldsymbol{X}_{t_{(1:m-1)}-t_m\mathbf{1}_{[m]}}\right).$$

In particular the expectation is constant and the covariance depends only on the modulus of the difference of the time points

$$\underline{\mathrm{Cum}}_2(\underline{X}_t, \underline{X}_{t+s}) = \left(\delta_{s\geq 0} I + \delta_{s<0} K^{-1}_{\mathfrak{p}_{2\to 1}}(d_{[2]})\right) \underline{\mathrm{Cum}}_2\left(\underline{X}_0, \underline{X}_{|s|}\right).$$

Notice the difference between the covariance $\underline{\underline{\mathrm{Cov}}}(\underline{X}_{t+s}, \underline{X}_t)$ and $\underline{\mathrm{Cum}}_2(\underline{X}_{t+s}, \underline{X}_t)$, i.e.,

$$\underline{\mathrm{Cum}}_2(\underline{X}_t, \underline{X}_{t+s}) = \mathrm{Vec}[\underline{\underline{\mathrm{Cov}}}(\underline{X}_{t+s}, \underline{X}_t)'].$$

Assumption 12. Besides the stationarity of kth order, we shall assume in the sequel that $\mathsf{E}\,\underline{X}_t = 0$ and

$$\sum_{\substack{t_j=-\infty,\\ j=1,2,\dots k-1}}^{\infty} \left\|\underline{\mathrm{Cum}}_k(\underline{X}_0, \boldsymbol{X}_{t_{(1:k-1)}})\right\| < \infty, \tag{14}$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector.

Assumption (14) implies that the Fourier transform of the cumulants exists is bounded and uniformly continuous. It will be convenient to use the notations $z = e^{i2\pi\omega}$ and $z_k = e^{i2\pi\omega_k}$ such that

$$z_{(1:k)}^{-t_{(1:k)}} = \exp\left\{-i2\pi \sum t_j\omega_j\right\}.$$

Definition 13. When it exists the Fourier transform

$$\underline{S}_k\left(\omega_{(1:k-1)}\right) = \sum_{\substack{t_j=-\infty,\\ j=1,2,\dots k-1}}^{\infty} \underline{\mathrm{Cum}}_k\left(\underline{X}_0, \boldsymbol{X}_{t_{(1:k-1)}}\right) z_{(1:k-1)}^{-t_{(1:k-1)}}, \tag{15}$$

of the cumulant of a kth order stationary process $\underline{X}_t$ is called a kth order Kronecker cumulant spectrum (or simply just a kth order spectrum).

In particular, the second order Kronecker spectrum is usually simply called a spectrum; it is the spectral density of the process because under assumption (14) the spectral measure $\mathrm{Vec}\,\underline{\underline{\Phi}}(d\omega)$ of the process is absolute continuous with respect to the Lebesgue measure and $\underline{S}_2$ is the corresponding Radon–Nikodym derivative of $\mathrm{Vec}\,\underline{\underline{\Phi}}'$.

Note here that the kth order Kronecker cumulant spectrum is *not* symmetric function of $k-1$ variables and the place of $\underline{X}_0$ in the $\underline{\mathrm{Cum}}_k$ is fixed. Consider a linear filter of the form

$$\underline{Y}_t = \sum_{m=-\infty}^{\infty} A_m \underline{X}_{t-m},$$

such that along with the process $\underline{X}_t$ the process $\underline{Y}_t$ is kth order stationary also. When it exists define the transfer function of this linear operator by

$$A(z) = \sum_{m=-\infty}^{\infty} A_m z^{-m},$$

then the spectrum $\underline{S}_{2,\underline{Y}}$ of $\underline{Y}_t$ is given by

$$\underline{S}_{2,\underline{Y}}(z) = \left[A(z^{-1}) \otimes A(z)\right] \underline{S}_{2,\underline{X}}(z).$$

In particular, when the $\underline{X}_t$ is independent and identically distributed (i.i.d.) all spectra of the process $\underline{X}_t$ are constant and they are $\underline{\mathrm{Cum}}_k(\boldsymbol{X}_{0_{[k]}})$. So the kth order spectrum of a linear process has the form

$$\underline{S}_{k,\underline{Y}}\left(z_{(1:k-1)}\right) = \left[A(z_k) \otimes \prod\nolimits_{m=1:k}^{\otimes} A(z_m)\right] \underline{\mathrm{Cum}}_k\left(\boldsymbol{X}_{0_{[k]}}\right). \tag{16}$$

where $z_k = z_{1:k-1}^{-\mathbf{1}_{[k-1]}}$. The higher order spectra can be obtained from the stochastic spectral measure $\underline{W}(d\omega)$ of the independent series $\underline{X}_t$ by the following way, see [2]. Let $\underline{S}_k(z_{(1:k-1)})$ be the kth order Kronecker spectra of $\underline{X}_t$, then

$$\underline{\mathrm{Cum}}_k(\underline{W}(d\omega_1), \underline{W}(d\omega_2), \dots, \underline{W}(d\omega_k)) = \delta_1\left(z_k = z_{1:k-1}^{-\mathbf{1}_{[k-1]}}\right) \underline{S}_k(z_{(1:k)})\, d\omega_{(1:k)},$$

where δ_1 is the Dirac comb. If one permutes the variables of $\underline{\mathrm{Cum}}_k$ the permutation goes on the spectra as well and because of the presence of the Dirac comb we get

$$\underline{S}_2(z_1) = \delta_1\left(z_2 = z_1^{-1}\right) K_{\mathfrak{p}_{2\to 1}}^{-1}\left(d_{[2]}\right) \underline{S}_2(z_2, z_1)$$

and therefore

$$\underline{S}_2(z) = K_{\mathfrak{p}_{2\to 1}}^{-1}\left(d_{[2]}\right) \underline{S}_2(z^{-1}),$$

i.e., the spectrum still can be considered on the interval $[0, 1/2]$. We shall frequently consider the third order spectrum $\underline{S}_3$ called the bispectrum as well, [1]. The bispectrum is generally complex valued. It has the following properties of symmetry,

$$\begin{aligned}
\underline{S}_3\left(z_{(1:2)}\right) &= \delta_1\left(z_1, z_2, z_3 = z_1^{-1} z_2^{-1}\right) K_{\mathfrak{p}_{2\to 1}}^{-1}\left(d_{[3]}\right) \underline{S}_3(z_{\mathfrak{p}_{2\to 1}(1:3)}) = K_{\mathfrak{p}_{2\to 1}}^{-1}\left(d_{[3]}\right) \underline{S}_3(z_2, z_1) \\
&= K_{\mathfrak{p}_{3\to 2}}^{-1}\left(d_{[3]}\right) \underline{S}_3(z_1, z_3) = K_{\mathfrak{p}_{3,1,2}}^{-1}\left(d_{[3]}\right) \underline{S}_3(z_3, z_1) = K_{\mathfrak{p}_{3,2,1}}^{-1}\left(d_{[3]}\right) \underline{S}_3(z_3, z_2) \\
&= K_{\mathfrak{p}_{2,3,1}}^{-1}\left(d_{[3]}\right) \underline{S}_3(z_2, z_3) = \overline{\underline{S}_3}\left(z_{(1:2)}^{-\mathbf{1}_{[2]}}\right) = \underline{S}_3(z_1 + k, z_2 + j), \\
&\qquad k, j = 1, 2, \dots .
\end{aligned}$$

These equations imply that there are twelve triangles of frequencies in the plane, each of which can be considered as the basic domain for the bispectrum. It is completely specified over the entire plane if it is determined over any one of the twelve triangles. We take the triangle with vertices $(0,0)$, $(1/2, 0)$, $(1/3, 1/3)$ as the basic domain for the bispectrum.

Let the process $\underline{X}_t$ be centered, therefore its third order cumulants are

$$\underline{\mathrm{Cum}}(k, l) = \mathsf{E}\, \underline{X}_t \otimes \underline{X}_{t+k} \otimes \underline{X}_{t+l}, \qquad k, l = 0, 1, 2, \dots .$$

The bispectrum according to the cumulants $\underline{\mathrm{Cum}}(k,l)$ is defined by

$$\underline{S}_3(z_1,z_2)=\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty}\underline{\mathrm{Cum}}(k,l)z_1^{-k}z_2^{-l}.$$

Put

$$\underline{S}_3^I(z_1,z_2)=\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\underline{\mathrm{Cum}}(k+l,l)z_1^{-(k+l)}z_2^{-l},\qquad \underline{S}_{\underline{X},\underline{X}^{\otimes 2}}(z)=\sum_{l=-\infty}^{\infty}\underline{\mathrm{Cum}}(l,0)z_1^{-l},$$

then we have a reasonable simple form for calculation in particular cases

$$\underline{S}_3(z_1,z_2)=\sum_{\mathfrak{p}\in\mathfrak{P}_3}K_{\mathfrak{p}(1:3)}\big(d_{[3]}\big)\underline{S}_3^I\left(z_{\mathfrak{p}(1)},z_{\mathfrak{p}(2)}\right)+\sum_{k=1:3}K_{\mathfrak{p}_{k,1}}\big(d_{[3]}\big)\underline{S}_{\underline{X},\underline{X}^{\otimes 2}}(z_k)-2\underline{\mathrm{Cum}}(0,0),$$

where $\mathfrak{p}(1:3)=(\mathfrak{p}(1),\mathfrak{p}(2),\mathfrak{p}(3))$, $\mathfrak{p}\in\mathfrak{P}_3$, $z_1=z_2^{-1}z_3^{-1}$and $K_{\mathfrak{p}_{1,1}}$ is the identity.

6. The multiple Wiener–Itô integral

A basic tool of investigations of nonlinear problems of time series analysis is the multiple Wiener–Itô integral, we refer to [20], [5], [6], [14] for details.

Let us define $\overline{L_\Phi^n}$, the real Hilbert space of complex valued functions defined on $\mathbf{R}^n$, according to a spectral measure Φ by the following:

- $f_n(\omega_1,\omega_2,\dots,\omega_n)=\overline{f_n(-\omega_1,-\omega_2,\dots,-\omega_n)}$,
- f_n is periodic with $\mod (1)$ in each variable:

$$f_n(\omega_1,\omega_2,\dots,\omega_k\pm 1,\dots,\omega_n)=f_n(\omega_1,\omega_2,\dots,\omega_k,\dots,\omega_n),\\ k=1,2,\dots,n.$$

- Now suppose that $\underline{\underline{\Phi}}(d\omega)$ is a matrix spectral measure $\underline{\underline{\Phi}}(d\omega)$ according to the vector stochastic spectral measure $\underline{W}(d\omega)$ with dimension d, and $\underline{f}_n$ denotes a vector with dimension d^n, such that each entry is from $\overline{L_\Phi^n}$, moreover

$$\|\underline{f}_n\|_{\underline{\underline{\Phi}}}^2=\int_{[0,1]^n}\left[\underline{f}_n(\omega_{(1:n)})\otimes\overline{\underline{f}_n(\omega_{(1:n)})}\right]'\prod\nolimits_{j=1:n}^{\otimes}\mathrm{Vec}\,\underline{\underline{\Phi}}(d\omega_i)<\infty.$$

The subspace $\widetilde{L_{\underline{\underline{\Phi}}}^n}$ consists of those functions $\underline{f}_n$ which are symmetric, i.e.,

- For every function $\underline{f}_n$ there corresponds an $\widetilde{\underline{f}_n}$ (symmetrized) version of $\underline{f}_n$, defined by

$$\begin{aligned}\widetilde{f_n}(\omega_1,\omega_2,\dots,\omega_n)&=\mathrm{sym}_{\mathfrak{p}}\,f_n(\omega_1,\omega_2,\dots,\omega_n)\\&=\frac{1}{n!}\sum_{\mathfrak{p}\in\mathfrak{P}_n}K_{\mathfrak{p}(1:n)}\left(d_{[n]}\right)f_n\left(\omega_{\mathfrak{p}(1)},\omega_{\mathfrak{p}(2)},\dots,\omega_{\mathfrak{p}(n)}\right).\end{aligned}$$

  summation is for all $\mathfrak{p}\in\mathfrak{P}_n$, where $\mathfrak{P}_n$ denotes the set of all permutations of the numbers $(1:n)=(1,2,\dots,n)$.

For each fixed n we shall define the stochastic integral, as a linear bounded (continuous) map from the Hilbert space $\overline{L_{\underline{\underline{\Phi}}}^n}$ into the Hilbert space of L_2-functionals of a stationary Gaussian process $\underline{X}_t$. L_2-functionals of a process $\underline{X}_t$ are those random variables which are measurable with respect to the σ-algebra generated by the process $\underline{X}_t$ and have a finite variance.

6.1. The multiple Wiener–Itô integral of second order. If $n = 1$, then the stochastic integral for $\underline{g} \in \overline{L^1_{\underline{\underline{\Phi}}}}$ is given by the formula

$$\mathtt{I}_1(\underline{g}) = \int_0^1 \underline{g}'(\omega)\, \underline{W}(d\omega).$$

$\mathtt{I}_1(\underline{g})$ is a real Gaussian variable and belongs to the Hilbert space $L^2_1(\mathcal{X})$ defined by all linear combination of X_t. Actually, $\mathtt{I}_1(\cdot)$ is an isometry from $\overline{L^1_{\underline{\underline{\Phi}}}}$ into $L^2_1(\mathcal{X})$, i.e.,

$$\left\|\underline{g}(\omega)\right\|^2_{\underline{\underline{\Phi}}} = \int_0^1 \left[\overline{\underline{g}(\omega)}' \otimes \underline{g}'(\omega)\right] \operatorname{Vec}(\underline{\underline{\Phi}}(d\omega)) = \left\|\mathtt{I}_1(\underline{g})\right\|^2 < \infty.$$

The linear transformation according to $\underline{g}(\omega)$ can be derived directly. Consideration now turns to the case $n = 2$. Let a Gaussian stochastic spectral measure $\underline{W}$ with spectral measure $\underline{\underline{\Phi}}$ be given. In the space, $\overline{L^2_{\underline{\underline{\Phi}}}}$ the subspace of all finite linear combination of functions

$$\left\{\underline{g}_{1k}(\omega_1) \otimes \underline{g}_{1m}(\omega_2), \underline{g}_{1k}, \underline{g}_{1m} \in \overline{L^1_{\underline{\underline{\Phi}}}}\right\}$$

is dense, i.e., every function $\underline{g}_2(\omega_1, \omega_2) \in \overline{L^2_{\underline{\underline{\Phi}}}}$ can be approximated in L_2 norm by some linear combination of the functions of this type. Therefore it is enough to define the stochastic integral for a linear combination of the functions of the type $\underline{g}_{1k}(\omega_1) \otimes \underline{g}_{1m}(\omega_2)$. In other words the integral

$$\mathtt{I}_2\left(\sum_{k,m=1}^{K,M} a_{km} \underline{g}_{1k} \otimes \underline{g}_{1m}\right)$$

is defined by

$$\begin{aligned}\mathtt{I}_2&\left(\sum_{k,m=1}^{K,M} a_{km} g_{1k}(\omega_1) \otimes g_{1m}(\omega_2)\right)\\ &= \sum_{k,m=1}^{K,M} a_{km} \underline{H}_2\left(\int_0^1 \underline{g}'_{1k}(\omega)\, \underline{W}(d\omega), \int_0^1 \underline{g}'_{1m}(\omega)\, \underline{W}(d\omega)\right),\end{aligned} \tag{17}$$

where $\underline{H}_2$ denotes the Hermite polynomial of second order. Note that it would be enough to define the integral above on the set of all finite linear combination of functions As we have seen, the Hermite polynomial $\underline{H}_2$ is bilinear, see Proposition 7, showing how the integral $\mathtt{I}_2$ follows for the linear combinations as well. Notice further that $\underline{H}_2$ is symmetrical, therefore by definition (17)

$$\mathtt{I}_2\big(\underline{g}_{11}(\omega_1) \otimes \underline{g}_{12}(\omega_2)\big) = \mathtt{I}_2\big(K_{\mathfrak{p}_{2\to 1}}\left[\underline{g}_{11}(\omega_2) \otimes \underline{g}_{12}(\omega_1)\right]\big) = \mathtt{I}_2\big(\operatorname{sym}_{\mathfrak{p}}\big(\underline{g}_{11}(\omega_1) \otimes \underline{g}_{12}(\omega_2)\big)\big).$$

Now we consider a particular permutation $\mathfrak{q}(1:n, \mathfrak{p}(1:n)) \in \mathfrak{P}_{2n}$ of the numbers $(1:2n)$. The permutation $\mathfrak{q}(1:n, \mathfrak{p}(1:n)) \in \mathfrak{P}_{2n}$is defined by the following way. For each $\mathfrak{p}(1:n) \in \mathfrak{P}_n$ fixed, $\mathfrak{q}(2k-1) = k$, $k = 1, \dots, n$, and $\mathfrak{q}(2k) = n + \mathfrak{p}(k)$, $k = 1, \dots, n$, i.e., $\mathfrak{q}(1:n, \mathfrak{p}(1:n)) = (1, \mathfrak{p}(1)+n, 2, \mathfrak{p}(2)+n, \dots, n, \mathfrak{p}(n)+n)$. For example, if $n = 3$ and $\mathfrak{p} = (3, 2, 1)$, then $\mathfrak{q}(1:3, \mathfrak{p}) = (1, 6, 2, 5, 3, 4)$. The permutation $\mathfrak{q}(1:n, \mathfrak{p}(1:n))$ might be reached in two steps, first take the permutation $(1:n, \mathfrak{p}(1:n)+n) \in \mathfrak{P}_{2n}$ then apply the permutation $(1, 1+n, 2, 2+n, \dots, n, 2n)$, actually that last one has been defined as $\mathfrak{q}(1:n, (1:n))$. The product of these two permutations gives $\mathfrak{q}(1:n, \mathfrak{p}(1:n))$, i.e.,

$$\mathfrak{q}(1:n, \mathfrak{p}(1:n)) = (1, 1+n, 2, 2+n, \dots, n, 2n) \times (1:n, \mathfrak{p}(1:n)+n). \tag{18}$$

The integral $\mathtt{I}_2$ has zero mean by the definition (17) and the variance of the integral follows from (19)

$$\begin{aligned}
&\mathsf{E}\,\mathtt{I}_2\big(\underline{g}_{11}\otimes\underline{g}_{12}\big)\mathtt{I}_2\big(\underline{f}_{11}\otimes\underline{f}_{12}\big)\\
&\quad=\int_{[0,1]^4}\Big(\overline{\underline{g}_{11}(\omega_1)}\otimes\overline{\underline{g}_{12}(\omega_2)}\otimes\underline{f}_{11}(\lambda_1)\underline{f}_{12}(\lambda_2)\Big)'\\
&\qquad\qquad\qquad\times\underline{\mathrm{Cum}}_2\Big(\underline{H}_2\Big[\overline{\underline{W}(d\omega_1)},\overline{\underline{W}(d\omega_2)}\Big],\underline{H}_2[\underline{W}(d\lambda_1),\underline{W}(d\lambda_2)]\Big)\\
&\quad=\int_{[0,1]^4}\Big(\overline{\underline{g}_{11}(\omega_1)}\otimes\overline{\underline{g}_{12}(\omega_2)}\otimes\underline{f}_{11}(\lambda_1)\otimes\underline{f}_{12}(\lambda_2)\Big)'\\
&\qquad\times\Big[\delta_1(\omega_1+\lambda_1)\delta_1(\omega_2+\lambda_2)K^{-1}_{\mathfrak{q}(1:2,(1:2))}(\underline{S}_2(\omega_1)\otimes\underline{S}_2(\omega_2))\\
&\qquad\quad+\delta_1(\omega_1+\lambda_2)\delta_1(\omega_2+\lambda_1)K^{-1}_{\mathfrak{q}(1:2,(2,1))}\big(\underline{S}_2(\omega_1)\otimes\underline{S}_2(\omega_2)\big)\Big]\,d\omega_1\,d\omega_2\,d\lambda_1\lambda_2\\
&\quad=2!\int_{[0,1]^2}\Big(\overline{\underline{g}_{11}(\omega_1)}\otimes\overline{\underline{g}_{12}(\omega_2)}\otimes\operatorname{sym}_{\mathfrak{p}}\Big[\underline{f}_{11}(\omega_1)\otimes\underline{f}_{12}(\omega_2)\Big]\Big)'\\
&\qquad\qquad\qquad\qquad\qquad\times K^{-1}_{\mathfrak{q}(1:2,(1:2))}[\underline{S}_2(\omega_1)\otimes\underline{S}_2(\omega_2)]\,d\omega_1\,d\omega_2,
\end{aligned}\tag{19}$$

where we considered that $\mathfrak{q}(1:2,(2,1))=(1,4,2,3)=(1,3,2,4)\times(1,2,4,3)$ therefore

$$K^{-1}_{\mathfrak{q}(1:2,(2,1))}=K^{-1}_{(1,2,4,3)}K^{-1}_{(1,3,2,4)}=K'_{(1,2,4,3)}K^{-1}_{\mathfrak{q}(1:2,(1:2))},$$

hence the result. For the particular case

$$\underline{g}_2(\omega_1,\omega_2)=\underline{g}_{11}(\omega_1)\otimes\underline{g}_{12}(\omega_2),$$

we have

$$\begin{aligned}
\mathsf{E}\,\mathtt{I}_2^2\big(\underline{g}_{11}\otimes\underline{g}_{12}\big)&=\int_{[0,1]}\Big(\overline{\underline{g}_{11}(\omega)}\otimes\underline{g}_{11}(\omega)\Big)'\underline{S}_2(\omega)\,d\omega=\big\|\underline{g}_{11}\big\|^2_{\underline{\underline{\Phi}}}\big\|\underline{g}_{12}\big\|^2_{\underline{\underline{\Phi}}}+(\underline{g}_{11},\underline{g}_{12})^2_{\underline{\underline{\Phi}}}\\
&=2!\,\Big\|\operatorname{sym}_{\mathfrak{p}}\big(\underline{g}_{11}\otimes\underline{g}_{12}\big)\Big\|^2_{\underline{\underline{\Phi}}}.
\end{aligned}$$

The basic properties of the stochastic integral of second order for $\underline{g}_2\in\overline{L^2_{\underline{\underline{\Phi}}}}$ may now be summarized as

1. $\mathtt{I}_2(\underline{g}_2)$ is real valued,
2. $\mathsf{E}\,\mathtt{I}_2(\underline{g}_2)=0$,
3. $\mathtt{I}_2(\underline{g}_2)=\mathtt{I}_2\big(\operatorname{sym}_{\mathfrak{p}}\underline{g}_2(\omega_1,\omega_2)\big)$,
4. $\mathsf{E}\,\mathtt{I}_2^2(\underline{g}_2)=2!\big\|\underline{g}_2\big\|_{\underline{\underline{\Phi}}}$ for $\underline{g}_2\in\widetilde{L^2_{\underline{\underline{\Phi}}}}$,
5. $\mathsf{E}\,\mathtt{I}_1(\underline{g}_1)\mathtt{I}_2(\underline{g}_2)=0$ for any $\underline{g}_1\in\overline{L^1_{\underline{\underline{\Phi}}}}$ and $\underline{g}_2\in\overline{L^2_{\underline{\underline{\Phi}}}}$,
6. if $\underline{g}_2,\underline{f}_2\in\overline{L^2_{\underline{\underline{\Phi}}}}$, then

$$\begin{aligned}
&\mathsf{E}\,\mathtt{I}_2(\underline{g}_2)\mathtt{I}_2(\underline{f}_2)\\
&\quad=2!\int_{[0,1]^2}\Big(\overline{\underline{g}_2(\omega_1,\omega_2)}\otimes\operatorname{sym}_{\mathfrak{p}}\underline{f}_2(\omega_1,\omega_2)\Big)'\\
&\qquad\qquad\qquad\qquad\times K^{-1}_{\mathfrak{q}(1:2,(1:2))}[\underline{S}_2(\omega_1)\otimes\underline{S}_2(\omega_2)]\,d\omega_1\,d\omega_2\\
&\quad=2!\int_{[0,1]^2}\Big(\overline{\operatorname{sym}_{\mathfrak{p}}\underline{g}_2(\omega_1,\omega_2)}\otimes\operatorname{sym}_{\mathfrak{p}}\underline{f}_2(\omega_1,\omega_2)\Big)'\\
&\qquad\qquad\qquad\qquad\times K^{-1}_{\mathfrak{q}(1:2,(1:2))}[\underline{S}_2(\omega_1)\otimes\underline{S}_2(\omega_2)]\,d\omega_1\,d\omega_2.
\end{aligned}$$

We shall use also the notation

$$\mathtt{I}_2(\underline{g}_2) = \int_{[0,1]^2} \underline{g}_2(\omega_1, \omega_2)\, \underline{W}(d\omega_1, d\omega_2).$$

6.1.1. The multiple Wiener–Itô integral of order n. We proceed with the general case. Functions of the space $\overline{L^n_{\underline{\underline{\Phi}}}}$ are approximated by linear combinations of the products $\prod^{\otimes}_{k_{(1:n)}} \underline{g}_{1k_{(1:n)}}$,where each $\underline{g}_{1k_j} \in \overline{L^1_{\underline{\underline{\Phi}}}}$. The multiple Wiener–Itô integral of order n, is defined by the equation

$$\begin{aligned}
\mathtt{I}_n &\left(\sum_{k_{(1:n)}=\mathbf{1}_{[n]}}^{K_{(1:n)}} a_{k_{(1:n)}} \prod^{\otimes}_{j=1:n} \underline{g}_{1k_j}(\omega_j) \right) \\
&= \sum_{k_{(1:n)}=\mathbf{1}_{[n]}}^{K_{(1:n)}} a_{k_{(1:n)}} \\
&\qquad \times H_n \left(\int_0^1 g'_{1k_1}(\omega)\, \underline{W}(d\omega), \int_0^1 g'_{1k_2}(\omega)\, \underline{W}(d\omega), \dots, \int_0^1 \underline{g}'_{1k_n}(\omega)'\, \underline{W}(d\omega) \right),
\end{aligned} \tag{20}$$

where H_n is the Hermite polynomial of order n, and $\mathbf{1}_{[n]}$ denotes a row vector having all ones in its coordinates. In particular, if we put

$$\prod^{\otimes}_{j=1:n} \underline{g}_{1k_j}(\omega_j) = \prod^{\otimes}_{j=1:n} \underline{c}_{1k_j} \exp\left\{ i2\pi \left(\sum_j k_j \omega_j \right) \right\},$$

we get

$$\begin{aligned}
\mathtt{I}_n \left(\prod^{\otimes}_{j=1:n} \underline{c}_{1k_j} \exp\left\{ i2\pi \left(\sum_j k_j \omega_j \right) \right\} \right) &= H_n \left(\underline{c}'_{1k_1} X_{k_1}, \underline{c}'_{1k_2} X_{k_2}, \dots, \underline{c}'_{1k_n} X_{k_n} \right) \\
&= \left(\prod^{\otimes}_{j=1:n} \underline{c}_{1k_j} \right)' \underline{H}_n \left(\boldsymbol{X}_{k_{(1:n)}} \right).
\end{aligned}$$

The notation

$$\mathtt{I}_n(\underline{f}_n) = \int_{[0,1]^n} \underline{f}'_n(\omega_1, \omega_2, \dots, \omega_n)\, \underline{W}(d\omega_1, d\omega_2, \dots, d\omega_n),$$

for the nth order integral will also be used.

Properties are summarized as follows. Let $\underline{g}_n \in \overline{L^n_{\underline{\underline{\Phi}}}}$, then

1. $\mathtt{I}_n(\underline{g}_n)$ is real valued,
2. $\mathsf{E}\,\mathtt{I}_n(\underline{g}_n) = 0$,
3. $\mathtt{I}_n(\underline{g}_n) = \mathtt{I}_n\big(\operatorname{sym}_{\mathfrak{p}} \underline{g}_n(\omega_{(1:n)})\big)$,
4. For $\underline{g}_n \in \widetilde{L^n_{\underline{\underline{\Phi}}}}$

$$\begin{aligned}
\mathsf{E}\,\mathtt{I}_n^2(\underline{g}_n) = n! \int_{[0,1]^n} &\left[\underline{g}_n(\omega_{(1:n)}) \otimes \overline{\underline{g}_n(\omega_{(1:n)})} \right]' \\
&\times K^{-1}_{\mathfrak{q}(1:n,(1:n))} \prod^{\otimes}_{j=1:n} \underline{S}_2(\omega_i)\, d\omega_{(1:n)},
\end{aligned}$$

5. If $m \neq n$, then $\mathsf{E}\,\mathtt{I}_m(\underline{g}_m)\mathtt{I}_n(\underline{g}_n) = 0$ for any $\underline{g}_m \in \overline{L^2_m}$ and $\underline{g}_n \in \overline{L^n_{\underline{\underline{\Phi}}}}$,

6. For $\underline{g}_n, \underline{h}_n \in \overline{L^n_{\underline{\underline{\Phi}}}}$

$$
\begin{aligned}
\mathsf{E}\,\mathtt{I}_n\big(\underline{g}_n\big)\mathtt{I}_n\big(\underline{h}_n\big) &= n! \int_{[0,1]^n} \Big[\operatorname{sym}_{\mathfrak{p}} \underline{g}_n(\omega_{(1:n)}) \otimes \overline{\operatorname{sym}_{\mathfrak{p}} \underline{h}_n(\omega_{(1:n)})}\Big]' \\
&\qquad\qquad \times K^{-1}_{\mathfrak{q}(1:n,(1:n))} \prod\nolimits_{j=1:n}^{\otimes} \underline{S}_2(\omega_i)\, d\omega_{(1:n)} \\
&= n! \int_{[0,1]^n} \Big[\underline{g}_n(\omega_{(1:n)}) \otimes \overline{\operatorname{sym}_{\mathfrak{p}} \underline{h}_n(\omega_{(1:n)})}\Big]' \\
&\qquad\qquad \times K^{-1}_{\mathfrak{q}(1:n,(1:n))} \prod\nolimits_{j=1:n}^{\otimes} \underline{S}_2(\omega_i)\, d\omega_{(1:n)}.
\end{aligned}
$$

6.1.2. Diagram formula. One of the basic rules of manipulating with multiple Wiener–Itô integral called Diagram formula. It originates from the product of Hermite polynomials. Take the product of the $\underline{H}_1$ and $\underline{H}_{n-1}$:

$$
\begin{aligned}
&\underline{H}_{n-1}\left(\boldsymbol{X}_{k_{(1:n-1)}}\right) \otimes \underline{X}_{k_n} \\
&\quad = \underline{H}_n\left(\boldsymbol{X}_{k_{(1:n)}}\right) + \sum_{j=1}^{n-1} K^{-1}_{\mathfrak{p}_{(n,j)\to(1,2)}}\left(d_{(1:n)}\right)\left[\operatorname{Vec}(C_{k_j,k_n}) \otimes \underline{H}_{n-2}\left(\boldsymbol{X}_{k_{(1:j-1,j+1:n-1)}}\right)\right],
\end{aligned}
$$

where

$$C_{k_j,k_n} = \operatorname{Cov}(X_{k_j}, X_{k_n}) = \operatorname{Cov}(X_{k_j-k_n}, X_0)$$

because of the stationarity of X_t. This may be rewritten by multiple integrals as

$$
\begin{aligned}
&\int_{[0,1]^{n-1}} \underline{g}'_{n-1} \exp\left\{i2\pi\left(\sum_{l=1}^{n-1} k_l\omega_l\right)\right\} \underline{W}\left(d\omega_{(1:n-1)}\right) \otimes \int_0^1 \underline{h}'_1 \exp\left\{i2\pi k_n\omega\right\} \underline{W}(d\omega) \\
&\quad = \int_{[0,1]^n} [\underline{g}'_{n-1} \otimes \underline{h}'_1] \exp\left\{i2\pi\left(\sum_{l=1}^{n} k_l\omega_l\right)\right\} \underline{W}(d\omega_{(1:n)}) \\
&\qquad + \sum_{j=1}^{n-1} \int_{[0,1]^{n-2}} \int_0^1 [\underline{g}'_{n-1} \otimes \underline{h}'_1] \exp\left\{i2\pi\left(\sum_{l=1}^{n-1} k_l\omega_l\right)\right\} \exp\{-i2\pi k_n\omega_j\} \\
&\qquad\qquad \times K^{-1}_{\mathfrak{p}_{(n,j)\to(1,2)}}(d_{(1:n)})[\operatorname{Vec} \underline{\underline{\Phi}}(d\omega_j) \otimes I_{d^{n-2}}] \\
&\qquad\qquad\qquad \times \underline{W}\left(d\omega_{(1:j-1)}, d\omega_{(j+1:n-1)}\right).
\end{aligned}
\tag{21}
$$

We introduce a function corresponding to the deterministic integral of the right hand side in (21). Let $\underline{h}_1 \in \overline{L^1_{\underline{\underline{\Phi}}}}$ and $\underline{g}_{n-1} \in \overline{L^{n-1}_{\underline{\underline{\Phi}}}}$ and define the function $\underline{g}_{n-1} \boxtimes_j \underline{h}_1$ of $(n-2)$ variables by

$$
\begin{aligned}
&\left[\underline{g}_{n-1} \boxtimes_j \underline{h}_1\right]\left(\omega_{(1:j-1)}, \omega_{(j+1:n-1)}\right) \\
&\quad = \int_0^1 \left[\underline{g}_{n-1}(\omega_{(1:n-1)}) \otimes \overline{\underline{h}_1(\omega_j)}\right]' K^{-1}_{\mathfrak{p}_{(n,j)\to(1,2)}} [\operatorname{Vec} \underline{\underline{\Phi}}(d\omega_j) \otimes I_{d^{n-2}}].
\end{aligned}
$$

It is easy to see that $\underline{g}_{n-1} \boxtimes_j \underline{h}_1 \in \overline{L^{n-2}_{\underline{\underline{\Phi}}}}$. The formula for the product $\mathtt{I}_{n-1}(\underline{g}_{n-1}) \otimes \mathtt{I}_1(\underline{h}_1)$ is now straightforward consequence of (21)

$$
\begin{aligned}
\mathtt{I}_{n-1}(\underline{g}_{n-1}) \otimes \mathtt{I}_1(\underline{h}_1) &= \int_{[0,1]^n} \left[\underline{g}_{n-1}(\omega_{(1:n-1)}) \otimes \underline{h}_1(\omega_n)\right]' \underline{W}(d\omega_{(1:n)}) \\
&\quad + \sum_{j=1}^{n-1} \int_{[0,1]^{n-2}} \left[\underline{g}_{n-1} \boxtimes_j \underline{h}_1\right]\left(\omega_{(1:j-1)}, \omega_{(j+1:n-1)}\right) \\
&\qquad\qquad \times \underline{W}(d\omega_{(1:j-1)}, d\omega_{(j+1:n-1)}).
\end{aligned}
\tag{22}
$$

It is seen that the chaotic representation, i.e. Dobrushin's Theorem for subordinated functionals Y_t of Gaussian stationary vector processes in space $\mathfrak{L}^2(\mathcal{X})$ is valid. If Y_t is stationary as well then it can be described by the representation

$$Y_t = \sum_{n=0}^{\infty} \int_{[0,1]^n} \exp\left\{ i2\pi t \sum \omega_{(1:n)} \right\} \underline{g}'_n \left(\omega_{(1:n)}\right) \underline{W} \left(d\omega_{(1:n)}\right), \tag{23}$$

where the transfer function $\underline{g} = (\underline{g}_0, \underline{g}_1, \underline{g}_2, \dots, \underline{g}_n, \dots) \in \operatorname{Exp}(\widetilde{L^2_{\underline{\underline{\Phi}}}})$ is essentially unique, see Major [14] for scalar case.

7. Appendix

A. Vec operator, commutation, and partitions.

Let A an $m \times n$ and B a $p \times q$ matrices the Kronecker or tensor product of A and B is defined by $A \otimes B = [a_{ij} B]$. The Vec operator on a matrix $A = [a_{ij}]_{i=1:m, j=1:n}$ of $m \times n$ acts as stacking the columns of the matrix one underneath the other therefore $\operatorname{Vec} A$ is an mn dimensional vector. If A, B, and C are matrices such that the product ABC is defined then

$$\operatorname{Vec}(ABC) = (C' \otimes A) \operatorname{Vec}(B). \tag{24}$$

The Kronecker product $\otimes$ has the advantage of having possibility of commuting of the factors by the help of some linear operator called commutation matrix, see [13] for details. We shall use it mostly in particular case of vectors. Start with a matrix A of $m \times n$, the vector $\operatorname{Vec} A'$ is a permutation of the vector $\operatorname{Vec} A$ therefore there exists a particular permutation matrix $K_{m \cdot n}$ of order $mn \times mn$, called *commutation matrix*, and $K_{m \cdot n}$ is defined by the equation

$$K_{m \cdot n} \operatorname{Vec} A = \operatorname{Vec} A'.$$

In that case if $\boldsymbol{a}$ is $m \times 1$ and $\boldsymbol{b}$ is $n \times 1$ then $K_{m \cdot n}(\boldsymbol{b} \otimes \boldsymbol{a}) = \boldsymbol{a} \otimes \boldsymbol{b}$, also

$$\operatorname{Vec}(A \otimes B) = (I_n \otimes K_{q \cdot m} \otimes I_p) \operatorname{Vec} A \otimes \operatorname{Vec} B. \tag{25}$$

The commutation matrix is orthogonal $K'_{m \cdot p} = K^{-1}_{m \cdot p} = K_{p \cdot m}$.

Now consider a set of vectors $(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)$ with dimensions $d_{(1:n)} = (d_1, d_2, \dots, d_n)$ respectively. Define the matrix

$$K_{j+1 \leftrightarrow j}(d_{(1:n)}) = \prod\nolimits_{i=1:j-1}^{\otimes} I_{d_i} \otimes K_{d_{j+1} \cdot d_j} \otimes \prod\nolimits_{i=j+2:n}^{\otimes} I_{d_i},$$

where $\prod_{i=1:j}^{\otimes}$ means the Kronecker product of the matrices indexed by

$$(1:j) = (1, 2, \dots, j).$$

Clearly

$$\begin{aligned} &K_{j+1 \leftrightarrow j}(d_{(1:n)}) \prod\nolimits_{i=1:n}^{\otimes} \boldsymbol{a}_i \\ &\quad = \prod\nolimits_{i=1:j-1}^{\otimes} (I_{d_i} \boldsymbol{a}_i) \otimes (K_{d_{j+1} \cdot d_j} (\boldsymbol{a}_j \otimes \boldsymbol{a}_{j+1})) \otimes \prod\nolimits_{i=j+2:n}^{\otimes} (I_{d_i} \boldsymbol{a}_i) \\ &\quad = \prod\nolimits_{i=1:j-1}^{\otimes} \boldsymbol{a}_i \otimes \boldsymbol{a}_{j+1} \otimes \boldsymbol{a}_j \otimes \prod\nolimits_{i=j+2:n}^{\otimes} \boldsymbol{a}_i. \end{aligned}$$

Therefore one is able to transpose (to change the order of) the elements $\boldsymbol{a}_j$ and $\boldsymbol{a}_{j+1}$ in a Kronecker product of vectors by the help of the matrix $K_{j \leftrightarrow j+1}(d_{(1:n)})$. In general $K'_{j \leftrightarrow j+1}(d_{(1:n)}) = K^{-1}_{j \leftrightarrow j+1}(d_{(1:n)})$ but $K_{j+1 \leftrightarrow j} \neq K_{j \leftrightarrow j+1}$ because of the dimensions d_{j+1} and d_j are not necessarily equal, if they are, then

$$K_{j+1 \leftrightarrow j} = K_{j \leftrightarrow j+1} = K^{-1}_{j \leftrightarrow j+1} = K'_{j \leftrightarrow j+1}.$$

We remind that $\mathfrak{P}_n$ denotes the set of all permutations of the numbers

$$(1:n)=(1,2,\dots,n),$$

if $\mathfrak{p}\in\mathfrak{P}_n$ then $\mathfrak{p}(1:n)=(\mathfrak{p}(1),\mathfrak{p}(2),\dots,\mathfrak{p}(n))$. From this follows that for each permutation $\mathfrak{p}(1:n)=(\mathfrak{p}(1),\mathfrak{p}(2),\dots,\mathfrak{p}(n))$, $\mathfrak{p}\in\mathfrak{P}_\mathfrak{n}$, there exists a matrix $K_{\mathfrak{p}(1:n)}(d_{(1:n)})$ such that

$$K_{\mathfrak{p}(1:n)}(d_{(1:n)})\prod\nolimits_{i=1:n}^{\otimes}\boldsymbol{a}_i=\prod\nolimits_{i=1:n}^{\otimes}\boldsymbol{a}_{\mathfrak{p}(i)}, \tag{26}$$

just because any permutation $\mathfrak{p}(1:n)$ can be get by the product of transpositions of neighbor elements. According to the inverse of the permutation $\mathfrak{p}(1:n)$ there exists the inverse $K_{\mathfrak{p}(1:n)}^{-1}(d_{(1:n)})$ of $K_{\mathfrak{p}(1:n)}(d_{(1:n)})$ as well. Note that the entries of $d_{(1:n)}$ are not necessary equals, they are the dimensions of the vectors $\boldsymbol{a}_i$, $i=1,2,\dots,n$, by the original order what is fixed. In particular transposing two elements only, j and k, in the product will be denoted by $K_{\mathfrak{p}_{j\leftrightarrow k}}(d_{(1:n)})$. The permutation when the element j is put in the place k is $\mathfrak{p}_{j\to k}$ and the corresponding operator is $K_{\mathfrak{p}_{j\to k}}$. It will not be confusing to use both notations $K_{j\to k}$ and $K_{\mathfrak{p}_{j\to k}}$ for the same operators.

A.1. Ordered partitions. Put $\mathcal{P}_n$ for the set of all partitions $\mathcal{K}$ of the numbers $(1:n)$. If $\mathcal{K}=\{\boldsymbol{b}_1,\boldsymbol{b}_2,\dots,\boldsymbol{b}_m\}$ where each block $\boldsymbol{b}_j\subset(1:n)$ then $|\mathcal{K}|=m$ denotes the *size* of $\mathcal{K}$. We define the following relation among the blocks $\boldsymbol{b}_j\in\mathcal{K}$, $\boldsymbol{b}_j\leq\boldsymbol{b}_k$ if

$$\sum_{l\in\boldsymbol{b}_j}2^{-l}\leq\sum_{l\in\boldsymbol{b}_k}2^{-l}. \tag{27}$$

Actually equality in (27) is possible if and only if $j=k$. The partition $\mathcal{K}$ will be considered as ordered if both, the elements of a block are ordered inside the block, and the blocks are ordered by the relation $\boldsymbol{b}_j\leq\boldsymbol{b}_k$ also. We suppose that all partitions $\mathcal{K}$ of $\mathcal{P}_n$ are ordered. Denote $\boldsymbol{\lambda}_{(1:M)}=[\underline{\lambda}_1,\underline{\lambda}_2,\dots,\underline{\lambda}_M]$ with dimensions $[d_1,d_2,\dots,d_M]$. In that case the differential operator $D_{\boldsymbol{\lambda}_{\boldsymbol{b}}}^{\otimes|\boldsymbol{b}|}$ (see bellow) is well defined because the vector $\boldsymbol{\lambda}_{\boldsymbol{b}}=[\underline{\lambda}_j,j\in\boldsymbol{b}]$ denotes an ordered subset of vectors $[\underline{\lambda}_1,\underline{\lambda}_2,\dots,\underline{\lambda}_M]$ corresponding the ordering in $\boldsymbol{b}$.

The ordered partition $\mathcal{K}$ defines a permutation $\mathfrak{p}(\mathcal{K})\in\mathfrak{P}_n$ which is the listing all the numbers of $(1:n)$ by ordering the blocks and numbers inside the blocs.

B. The operator $D_{\boldsymbol{\lambda}_{(1:n)}}^{\otimes n}$.

Suppose that $\underline{\lambda}\in\mathbf{R}^d$, and the function $\underline{\phi}(\underline{\lambda})=(\phi_1(\underline{\lambda}),\phi_2(\underline{\lambda}),\dots,\phi_m(\underline{\lambda}))'$ is vector valued and differentiable k times. The Jacobian is denoted by

$$\frac{\partial}{\partial\underline{\lambda}'}=\left(\frac{\partial}{\partial\lambda_1},\frac{\partial}{\partial\lambda_2},\dots,\frac{\partial}{\partial\lambda_d}\right)$$

and the operator $D_{\underline{\lambda}}^{\otimes}$ defined as follows

$$D_{\underline{\lambda}}^{\otimes}\underline{\phi}=\operatorname{Vec}\left(\frac{\partial\underline{\phi}}{\partial\underline{\lambda}'}\right)',$$

formally $D_{\underline{\lambda}}^{\otimes}$ is the following Kronecker product

$$D_{\underline{\lambda}}^{\otimes}\underline{\phi}=(\phi_1(\underline{\lambda}),\phi_2(\underline{\lambda}),\dots,\phi_m(\underline{\lambda}))'\otimes\left(\frac{\partial}{\partial\lambda_1},\frac{\partial}{\partial\lambda_2},\dots,\frac{\partial}{\partial\lambda_d}\right)'.$$

Applying the operator $D_{\underline{\lambda}}^{\otimes k}$ to $\underline{\phi}$ means

$$D_{\underline{\lambda}}^{\otimes k}\underline{\phi}=D_{\underline{\lambda}}^{\otimes}\left(D_{\underline{\lambda}}^{\otimes k-1}\underline{\phi}\right)=(\phi_1(\underline{\lambda}),\phi_2(\underline{\lambda}),\dots,\phi_m(\underline{\lambda}))'\otimes\left(\frac{\partial}{\partial\lambda_1},\frac{\partial}{\partial\lambda_2},\dots,\frac{\partial}{\partial\lambda_d}\right)^{\prime\otimes k}$$

and the result is a column vector of order md^k, (might be called K-derivative), containing all the possible partial derivatives of entries of $\underline{\phi}$ according to the Kronecker product

$$\left(\frac{\partial}{\partial\lambda_1}, \frac{\partial}{\partial\lambda_2}, \dots, \frac{\partial}{\partial\lambda_d}\right)^{\prime\otimes k}.$$

We proceed with some properties of the operator $D^{\otimes}_{\underline{\lambda}}$. Put $\underline{\psi}(\underline{\lambda}) = A\underline{\phi}(\underline{\lambda})$, where A is $n \times m$, then if one use the preceding definition obtains

$$D^{\otimes}_{\underline{\lambda}}\underline{\psi} = (A \otimes I_d) D^{\otimes}_{\underline{\lambda}}\underline{\phi},$$

where I_d is the unite matrix with dimension d. Moreover

$$D^{\otimes}_{\underline{\lambda}}\underline{\phi}(A\underline{\lambda}) = (I_m \otimes A')\, D^{\otimes}_{\underline{\mu}}\underline{\phi}(\underline{\mu})\Big|_{\underline{\mu}=A\underline{\lambda}}.$$

In particular, if $m = 1$, i.e., ϕ is scalar then

$$D^{\otimes}_{\underline{\lambda}}\phi(A\underline{\lambda}) = A'\, D^{\otimes}_{\underline{\mu}}\phi(\underline{\mu})\Big|_{\underline{\mu}=A\underline{\lambda}}.$$

Another property of $D^{\otimes}_{\underline{\lambda}}$ is that if $\underline{\lambda} \in \mathbf{R}^d$, $\underline{\phi}_k \in \mathbf{R}^{m_k}$, $k = 1, 2, \dots, M$, then

$$D^{\otimes}_{\underline{\lambda}} \prod\nolimits^{\otimes}_{(1:M)} \underline{\phi}_k = \sum_{j=1}^{M} K^{-1}_{\mathfrak{p}_{M\to j}}(m_{1:M}, d) \left[\prod\nolimits^{\otimes}_{(1:j-1)} \underline{\phi}_k \otimes \left[D^{\otimes}_{\underline{\lambda}}\underline{\phi}_j(\underline{\lambda})\right] \otimes \prod\nolimits^{\otimes}_{(j+1:M)} \underline{\phi}_k\right].$$

An interesting case is obtaining the following

$$D^{\otimes k}_{\underline{\lambda}} \underline{x}^{\prime\otimes k} \underline{\lambda}^{\otimes k} = k!\, \underline{x}^{\otimes k}, \tag{28}$$

where $\underline{x}$ is a vector of dimension d. The reason of (28) is that the Kronecker product $\underline{x}^{\prime\otimes k}$ is invariant under permutation of its component vectors $\underline{x}$, i.e.,

$$\underline{x}^{\prime\otimes l} K_{j+1\to l}(d_{[l]}) = \underline{x}^{\prime\otimes l},$$

for any l and j.

We define the operator $D^{\otimes n}_{\boldsymbol{\lambda}_{(1:n)}}$ recursively, more precisely put $\boldsymbol{\lambda}_{(1:n)} = [\underline{\lambda}_1, \underline{\lambda}_2, \dots, \underline{\lambda}_n]$ with dimensions $[d_1, d_2, \dots, d_n]$, and

$$D^{\otimes}_{\underline{\lambda}_j}\underline{\varphi} = \operatorname{Vec}\left(\underline{\varphi}\frac{\partial}{\partial\underline{\lambda}'_j}\right)', \qquad \underline{\varphi} \in \mathbf{R}^d,$$

finally

$$D^{\otimes n}_{\boldsymbol{\lambda}_{(1:n)}}\underline{\varphi} = D^{\otimes}_{\underline{\lambda}_n}\big(D^{\otimes n-1}_{\boldsymbol{\lambda}_{(1:n-1)}}\underline{\varphi}\big),$$

is column vector of the partial differential operator of order n and first order by each variable $\underline{\lambda}_j$. The dimension of $D^{\otimes n}_{\boldsymbol{\lambda}_{(1:n)}}$ is $d_{1:n}^{\mathbf{1}_{[n]}} = \prod_{j=1}^{n} d_j$, where $\mathbf{1}_{[n]}$ denotes a row vector having all ones in its coordinates, i.e., $\mathbf{1}_{[n]} = (1, 1, \dots, 1)$ with dimension n. For a permutation $\mathfrak{p}$ of $(1 : n)$ we have

$$\left(I_d \otimes K^{-1}_{\mathfrak{p}(1:n)}(d_{(1:n)})\right) D^{\otimes n}_{\boldsymbol{\lambda}_{\mathfrak{p}(1:n)}}\underline{\varphi} = D^{\otimes n}_{\boldsymbol{\lambda}_{(1:n)}}\underline{\varphi}.$$

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